Master Thesis

Maximal Operators and Differentiation of Integrals

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Abstract

The following Master's thesis is an introduction to the theory of maximal operators and the differentiation of integrals. In the first section, we introduce the basic notions of differentiation, the maximal operator and important results such as the Sawyer-Stein principles. We also elucidate the deep connection between the covering properties of a basis and its differentiation properties. We motivate and illustrate all these notions with the example of the Hardy-Littlewood maximal operator. In the second section, we focus on the case of rectangles whose sides are parallel to the coordinate axes. We begin by studying the strong maximal operator and its generalizations, then we establish an important theorem due to Stokolos, which gives us a complete characterization of rectangles whose sides are parallel to coordinate axes in the plane. We conclude by introducing the well-known Zygmund conjecture. The final section concerns sloping rectangles in the plane. We introduce the construction of Perron trees and study Bateman's theorem and its consequences. We conclude with the almost-orthogonality principle, and give some potential applications for an open problem.

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Contents

1. General theory

1.1. Introduction and definitions

Definition 1.1.
- A **differentiation basis** of a point $x \in \mathbb{R}^n$ is a collection $\mathcal{B}(x)$ of open bounded sets containing x and having positive Lebesgue measure, such that

$$\exists\, (R_k)_k \subset \mathcal{B}(x), \quad \operatorname{diam}(R_k) \underset{k\to\infty}{\longrightarrow} 0.$$

- A map $\mathcal{B}$ defined on $\mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$, $\mathcal{B}(x)$ is a differentiation basis at x, is a differentiation basis in $\mathbb{R}^n$.

In the following, we'll just speak of a "diferentiation basis".

Examples 1.2.
- Let $\mathcal{Q}_n(x)$ be the set of all cubes of $\mathbb{R}^n$ containing x, then the map $\mathcal{Q}_n : x \longmapsto \mathcal{Q}_n(x)$ is a differentiation basis.
- Let $\mathcal{I}_n(x)$ be the set of all rectangles whose sides are parallel to the coordinate axes of $\mathbb{R}^n$ containing x. The map $\mathcal{I}_n : x \longmapsto \mathcal{I}_n(x)$ is a differentiation basis.
- Let $\mathcal{R}_n(x)$ be the set of all rectangles of $\mathbb{R}^n$ containing x. The map $\mathcal{R}_n : x \longmapsto \mathcal{R}_n(x)$ is a differentiation basis.
- We denote $\mathcal{Q}_n^*(x)$ the set of all cubes in $\mathbb{R}^n$ containing x and centered at x, and $\mathcal{Q}_n^*$ is the associated differentiation basis. We define the bases $\mathcal{I}_n^*$ and $\mathcal{R}_n^*$ as well.

Remark 1.3. We defined $\mathcal{R}_n$ the collection of all rectangles in $\mathbb{R}^n$ by

$$\mathcal{R}_n := \{\phi(R) : R \in \mathcal{I}_n, \phi \in \operatorname{Iso}(\mathbb{R}^n)\}$$

where $\operatorname{Iso}(\mathbb{R}^n)$ is the set of all the isometries of $\mathbb{R}^n$.

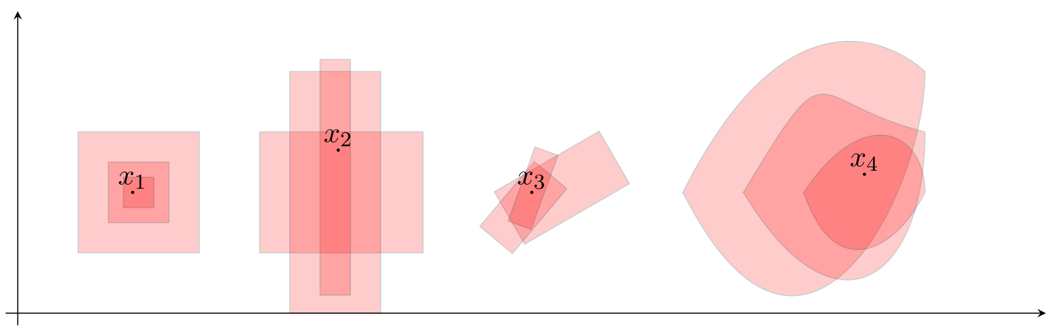

Figure 1: The basis $\mathcal{Q}_2, \mathcal{I}_2, \mathcal{R}_2$ and another basis in $\mathbb{R}^2$.

When there is no ambiguity with the dimension, we will just denote $\mathcal{Q}, \mathcal{I}, \mathcal{R}$.

Remark 1.4. We can also consider the basis composed of balls of $\mathbb{R}^n$. Because a ball is contained in a cube and vice versa, often we can understand the case of balls with cubes and conversely.

Most of our results will concern specific categories of bases, the Busemann-Feller bases, the translation invariant bases, and the homothecy invariant bases.

Definition 1.5. Let $\mathcal{B}$ be a basis. We say that

- $\mathcal{B}$ is **translation invariant** if for each $x \in \mathbb{R}^n$, one has

$$\mathcal{B}(x) = \{x + B : B \in \mathcal{B}(0)\}.$$

- $\mathcal{B}$ is a **Busemann-Feller** basis if for each $R \in \mathcal{B}$, $x \in R$, one has $R \in \mathcal{B}(x)$.
- $\mathcal{B}$ is **homothecy invariant** if for each $R \in \mathcal{B}$, any set homothetic to R with any ratio and any center is also in $\mathcal{B}$.

Example 1.6. The bases $\mathcal{Q}^*, \mathcal{I}^*, \mathcal{R}^*$ aren't Busemann-Feller bases, whereas $\mathcal{Q}, \mathcal{I}, \mathcal{R}$ are.

As its name suggests, a differentiation basis will be used to differentiate integrals. For f a locally summable function, $\mathcal{B}$ a basis and $x \in \mathbb{R}^n$, we write

$$\overline{D}_{\mathcal{B}}(f,x) := \sup_{\substack{(R_k)_k \subset \mathcal{B}(x) \\ \operatorname{diam}(R_k) \to 0}} \limsup_{k \to \infty} \frac{1}{|R_k|} \int_{R_k} f(y) \mathrm{d}y$$

and

$$\underline{D}_{\mathcal{B}}(f,x) := \inf_{\substack{(R_k)_k \subset \mathcal{B}(x) \\ \operatorname{diam}(R_k) \to 0}} \liminf_{k \to \infty} \frac{1}{|R_k|} \int_{R_k} f(y) \mathrm{d}y.$$

Definition 1.7. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathcal{B}$ a basis. We say that $\mathcal{B}$ differentiates the integral of f if

$$f(x) = \underline{D}_{\mathcal{B}}(f,x) = \overline{D}_{\mathcal{B}}(f,x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Obviously, if f is at least continuous, then every basis differentiate its integral. Instead of assume regularity with our functions, the challenge is to know whether we can differentiate the integral of a function with only integrability-type hypothesis, for example if $f \in L^1(\mathbb{R}^n)$ or if $f \in L^p(\mathbb{R}^n)$. If $f \in L^1(\mathbb{R}^n)$, there is the well-known Lebesgue differentiation theorem.

Theorem 1.8 (Lebesgue differentiation theorem, 1910)**.** If $f \in L^1(\mathbb{R}^n)$, then

$$f(x) = \lim_{r \to 0} \frac{1}{|B_r|} \int_{B_r(x)} f(y) \mathrm{d}y = f(x) \quad \text{for a.e } x \in \mathbb{R}^n,$$

where B_r denote the ball of radius r centered at the origin and $B_r(x)$ the ball of radius r centered at x.

We will see later that the Lebesgue spaces are too restrictive to handle all the situations. We will need a generalization of it, the Orlicz spaces $L^\Phi(\mathbb{R}^n)$(see the appendix A for definitions). It's also a suitable framework to define a notion of optimal differentiation space.

Definition 1.9. Let $\mathcal{B}$ a basis and $L^\Phi(\mathbb{R}^n)$ an Orlicz space.

- If $\mathcal{B}$ differentiates the integral of every function in $L^\Phi(\mathbb{R}^n)$, we say that $\mathcal{B}$ differentiates $L^\Phi(\mathbb{R}^n)$.
- If for every function g on $\mathbb{R}_+$ such that $\lim_{t\to\infty} g(t) = 0$ and $g \cdot \Phi$ is an Orlicz function, there exists an $f \in L^{g\cdot\Phi}(\mathbb{R}^n)$ such that

$$\sup_{\substack{(R_k)_k \subset \mathcal{B}(x)\\ \operatorname{diam}(R_k)\to 0}} \limsup_{k\to\infty} \frac{1}{|R_k|}\int_{R_k} f(y)\mathrm{d}y = \infty$$

  on a set of strictly positive measure, we say that $\mathcal{B}$ does not differentiates $o(L^\Phi(\mathbb{R}^n))$.

If $\mathcal{B}$ differentiates $L^\Phi(\mathbb{R}^n)$ and does not differentiates $o(L^\Phi(\mathbb{R}^n))$, we say that $\mathcal{B}$ differentiates precisely $L^\Phi(\mathbb{R}^n)$.

Locally, the space $L^\infty(\mathbb{R}^n)$ is very small. Indeed, for example we locally have $L^\infty(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for each $1 \leqslant p \leqslant \infty$. Differentiate $L^\infty(\mathbb{R}^n)$ can be seen as the weakest differentiation property for a basis. This motivates the following definition.

Definition 1.10. A basis $\mathcal{B}$ is a **density basis** if $\mathcal{B}$ differentiates $L^\infty(\mathbb{R}^n)$.

Remark 1.11. It's possible to define density bases in another equivalent way. A basis $\mathcal{B}$ is a density basis if and only if for each measurable set A, for almost every $x \in \mathbb{R}^n$, we have

$$\lim_{k\to\infty} \frac{|A\cap R_k|}{|R_k|} = \chi_A(x),$$

where $(R_k)_k$ is any arbitrary sequence of $\mathcal{B}(x)$ contracting to x. M. de Guzmán used this definition in [10], and proved the equivalence between the two statements.

Our main interest will be to answer these questions, at least partially.

- Given a basis $\mathcal{B}$, what class(es) of functions does $\mathcal{B}$ differentiates ?
- Conversely, given an Orlicz space $L^\Phi(\mathbb{R}^n)$, which bases differentiate $L^\Phi(\mathbb{R}^n)$?

1.2. **Maximal operators**

When a basis $\mathcal{B}$ is invariant by translations, the right tool to deal with these problems is the maximal operator associated to a basis. If R is a bounded Borel set and f a function locally integrable, let's denote

$$A_R f := \frac{1}{|R|}\int_R f$$

the average of f on R.

Definition 1.12. Let $\mathcal{B}$ be a basis. The **maximal operator** $M_\mathcal{B}$ associated to $\mathcal{B}$ is defined by

$$M_\mathcal{B} f(x) := \sup_{R\in\mathcal{B}(x)} A_R|f| = \sup_{R\in\mathcal{B}(x)} \frac{1}{|R|}\int_R |f|,$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$.

In other words, the maximal operator of a function f at x is the supremum of the averages of $|f|$ on the elements of the basis $\mathcal{B}(x)$.

Examples 1.13.
- The maximal operator associated to the basis $\mathcal{Q}^*$ is called the **Hardy-Littlewood** maximal operator, and noted M_{HL}. One has

$$M_{\mathrm{HL}}f(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} |f|.$$

This operator is fundamental in harmonic analysis, especially in the study of approximations of the identity. We can consult [11, Proposition 2.7, p31] for more information. The maximal operator associated to the cubes $\mathcal{Q}$ is known as the uncentered Hardy-Littlewood maximal operator, noted $\widetilde{M}_{\mathrm{HL}}$. These two maximal operators are equivalent in a certain way. One has

$$M_{\mathrm{HL}}f(x) \leqslant \widetilde{M}_{\mathrm{HL}}f(x) \leqslant 2^n M_{\mathrm{HL}}f(x)$$

for each $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$.

- $M_{\mathcal{I}}$ is know as the **strong maximal operator**, and noted M_{s}. Its case will be studied in section 2.1, it's the first natural case of a basis which doesn't differentiates $L^1(\mathbb{R}^n)$.

- Given $N > 1$, let $\mathcal{K}$ be the basis composed by all the rectangles in $\mathbb{R}^n$ with $n-1$ sides of length $h > 0$ and one side of the length Nh. We obtain the **Kakeya maximal operator**. There are important conjectures about this function, conjectures linked to the Hausdorff dimension of the Kakeya set. See [11, p47] for more information.

In general, we don't have an explicit description of $\mathcal{B}$, we sometimes simply prove the existence of a base with specific properties, without describing it precisely, see for example [25, Theorem 2]. Thus, the measurability of $M_{\mathcal{B}}f$ can be difficult to verify, even with simple bases. Nevertheless, everything goes well for a Busemann-Feller basis. We neither have any problem for the bases $\mathcal{Q}^*, \mathcal{I}^*, \mathcal{R}^*$.

Proposition 1.14. If $\mathcal{B}$ is Busemann-Feller basis, then

$$M_{\mathcal{B}} : L^1_{\mathrm{loc}}(\mathbb{R}^n) \longrightarrow \mathcal{M}(\mathbb{R}^n), \quad M_{\mathcal{B}} : L^\infty(\mathbb{R}^n) \longrightarrow L^\infty(\mathbb{R}^n).$$

Furthermore, $\|M_{\mathcal{B}}f\|_\infty \leqslant \|f\|_\infty$ whenever $f \in L^\infty(\mathbb{R}^n)$.

Proof. If $\mathcal{B}$ is Busemann-Feller basis, then for any $\lambda > 0$, we can easily check that the sets $\{M_{\mathcal{B}}f > \lambda\}$ are open. The second part is trivial. □

In this paper, we suppose that all the basis we'll be considering will carry locally integrable functions to measurable functions. If it's not the case, certain of the results of the theory can be stated with the exterior Lebesgue measure instead of the casual normal measure. The following shows the relationship between $M_{\mathcal{B}}$ and its differentiation properties.

Theorem 1.15. Let $\mathcal{B}$ be basis and $1 \leqslant p < \infty$. Assume that $M_{\mathcal{B}}$ is of weak-type (p,p), *i.e.* for each $f \in L^p(\mathbb{R}^n)$ and all $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}}f(x) > \lambda\}| \lesssim \left(\frac{\|f\|_p}{\lambda}\right)^p.$$

Then $\mathcal{B}$ differentiates $L^p(\mathbb{R}^n)$, *i.e.*

$$f(x) = \underline{D}_{\mathcal{B}}(f,x) = \overline{D}_{\mathcal{B}}(f,x) \quad \text{for a.e. } x \in \mathbb{R}^n,$$

for each $f \in L^p(\mathbb{R}^n)$.

Proof. Let $f \in L^p(\mathbb{R}^n)$ be fixed and $\varepsilon > 0$. For $\lambda > 0$, we define

$$\overline{P}_\lambda := \{x \in \mathbb{R}^n : |\overline{D}_{\mathcal{B}}(f,x) - f(x)| > \lambda\}$$

and

$$\underline{P}_\lambda := \{x \in \mathbb{R}^n : |\underline{D}_{\mathcal{B}}(f,x) - f(x)| > \lambda\}.$$

Our goal is to show that $|\underline{P}_\lambda| = |\overline{P}_\lambda| = 0$ for all $\lambda > 0$. By density, there exists a continuous function $g \in \mathcal{C}^0(\mathbb{R}^n)$ such that

$$\|f - g\|_p < \varepsilon.$$

Clearly, $\overline{D}_{\mathcal{B}}(f,x) \leqslant M_{\mathcal{B}}f(x)$ for almost every $x \in \mathbb{R}^n$ and $\overline{D}_{\mathcal{B}}(g,x) = g(x)$ for every $x \in \mathbb{R}^n$. So by writting $f = g + f - g$, one has

$$\begin{aligned} |\overline{P}_\lambda| &= |\{x \in \mathbb{R}^n : |\overline{D}_{\mathcal{B}}(f-g,x) + (g-f)(x)| > \lambda\}| \\ &\leqslant \left|\left\{x \in \mathbb{R}^n : |\overline{D}_{\mathcal{B}}(f-g,x)| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R}^n : |(g-f)(x)| > \frac{\lambda}{2}\right\}\right|. \end{aligned}$$

Since

$$\overline{D}_{\mathcal{B}}(f-g,x) \leqslant M_{\mathcal{B}}(f-g)(x),$$

by using the weak-type of $M_{\mathcal{B}}$ and Markov's inequality we have

$$|\overline{P}_\lambda| \leqslant \left(\frac{2C}{\lambda}\|f-g\|_p\right)^p + \left(\frac{2}{\lambda}\|f-g\|_p\right)^p \leqslant \left(\frac{2C+2}{\lambda}\right)^p \varepsilon^p.$$

where $C > 0$ is a constant. Since ε is arbitrary, we have $|\overline{P}_\lambda| = 0$. We proceed the same for $\underline{P}_\lambda$. ☐

Remark 1.16. The same result is true for Orlicz space. If $M_{\mathcal{B}}$ is of weak-type (Φ, Φ) where Φ is an Orlicz function, then $\mathcal{B}$ differentiates $L^\Phi(\mathbb{R}^n)$.

Let's try to apply this theorem to the Hardy-Littlewood maximal operator, in order to prove the theorem 1.8. We have to show that M_{HL} is of weak-type $(1,1)$. This result is based on a covering lemma.

Lemma 1.17. Let A be a finited reunion of open cubes,

$$A = \bigcup_{i=1}^{N} Q_{r_i}(x_i),$$

where $Q_{r_i}(x_i)$ designs the cubes centered at x_i of diameter r_i. Then, there exists a subset $I \subset \{1, \ldots, N\}$ such that

(i) the cubes $Q_{\frac{3}{2}r_i}(x_i)$ for $i \in S$ are disjoint ;

(ii) $A \subset \bigcup_{i \in I} Q_{r_i}(x_i)$;

(iii) $|A| \lesssim_n \sum_{i \in I} |Q_{r_i}(x_i)|$.

Proof. To simplify, let's note $Q_i := Q_{r_i}(x_i)$. We arrange the balls by decreasing diameter, we can therefore assume that

$$r_1 \geqslant r_2 \geqslant \cdots \geqslant r_N.$$

Let $i_1 := 1$. We note

$$J_1 := \{j \in [\![i_1 + 1, N]\!] : Q_j \cap Q_{i_1} \neq \emptyset\}$$

(we remove all the cubes which intersect Q_{i_1}) and $i_2 := \min J_1$. We continue with

$$J_2 := \{j \in [\![i_2+1, N]\!] : Q_j \cap Q_{i_2} \neq \emptyset\} \backslash J_1, \quad i_2 := \min J_2$$

and continue with $J_3, J_4, \ldots$. We stop when there are no more cubes to remove, it necessarily happens after a finite number of steps. Then, define I as $I := \{i_1, i_2, \ldots\} \subset [\![1, N]\!]$. The construction is illustrated in the figure 2, where we remove the grey cubes.

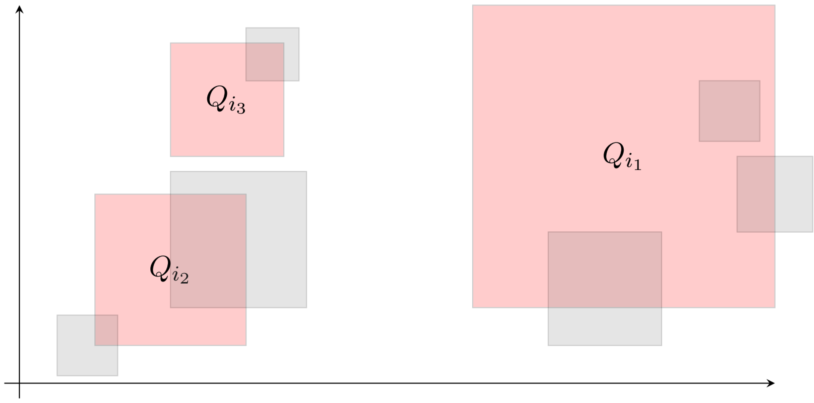

Figure 2: Illustration of lemma 1.17

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By construction, the cubes $(Q_i)_{i\in I}$ are disjoint. If $j \notin I$, it means that the cube Q_j intersects some Q_i for some $i \in I$ with $i \leqslant j$. But it's easy to see that in this case we have $Q_j \subset Q_{\frac{3}{2}r_i}(x_i)$. Indeed, if $x \in Q_j$ and $y \in Q_j \cap Q_i$, then

$$\|x - x_i\|_\infty \leqslant \|x - x_j\|_\infty + \|x_j - y\|_\infty + \|y - x_i\|_\infty \leqslant \frac{r_j}{2} + \frac{r_j}{2} + \frac{r_i}{2} \leqslant \frac{3}{2} r_i.$$

The second item is then proved, and it also directly implies the third and last item. □

We are now able to prove the Hardy-Littlewood maximal theorem.

Theorem 1.18 (Hardy-Littlewood maximal theorem)**.** The Hardy-Littlwood maximal operator M_{HL} is of weak-type $(1,1)$.

Remark 1.19. In fact, a weak-type $(1,1)$ for M_{HL} is the best we could hope for, we can't have a strong-type inequality for M_{HL}. Indeed, $M_{\mathrm{HL}}f$ is in $L^1(\mathbb{R}^n)$ if and only if f is identically 0 (see [11, p36]). Nevertheless, we can show that if f is an integrable function supported on a compact set K, then $M_{\mathrm{HL}} \in L^1(K)$ if and only if $f \in L \log^+ L(K)$ (see for example [22]).

Let's move to the proof.

Proof. Let $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$. We write $A = \{x \in \mathbb{R}^n : M_{\mathrm{HL}}f(x) > \lambda\}$. For each $x \in A$, there exists a cube $Q(x)$ centered at x such that

$$\int_{Q(x)} |f| > \lambda |Q(x)|.$$

These cubes cover A, $A \subset \cup_{x\in A} Q(x)$. Let $K \subset A$ be a compact subset of A. Then there are N points $x_1, \ldots, x_N$ such that

$$K \subset \bigcup_{i=1}^{N} Q(x_i).$$

By the lemma 1.17, there exists $I \subset \{1, \ldots, N\}$ such that

$$|K| \lesssim \sum_{i \in I} |Q(x_i)| \lesssim \frac{1}{\lambda} \int_{\cup_{i=1}^N Q(x_i)} |f| \leqslant \frac{\|f\|_1}{\lambda}.$$

Since the right term is independant of K, we can conclude by using the regularity of the Lebesgue measure. ☐

So, by theorem 1.15, the Lebesgue differentiation theroem is proven. By using the well known Marcinkiewicz interpolation theorem (see [11, Theorem 2.4, p29]), M_{HL} is even of strong-type (p, p) for $p > 1$:

$$M_{\mathrm{HL}} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \quad \text{and} \quad \|M_{\mathrm{HL}} f\|_p \lesssim \|f\|_p$$

where $f \in L^p(\mathbb{R}^n)$. Since a strong-type implies the corresponding weak-type, we also can assert that $\mathcal{Q}^*$ differentiates $L^p(\mathbb{R}^n)$, $1 < p < \infty$. We also have an analogue of the theorem 1.15 for the space $L^\infty(\mathbb{R}^n)$ and the density basis. See [14, Lemma 1] for the proof.

Theorem 1.20. Let $\mathcal{B}$ be a Busemann-Feller basis invariant by translations. Suppose that for any $\alpha \in (0, 1)$ there exists $r > 0$ and a constant $C < \infty$ such that for each mesurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$, we have

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}}^r \chi_E(x) > \alpha\}| \leqslant C|E|$$

where

$$M_{\mathcal{B}}^r \chi_E(x) := \sup_{\substack{R \in \mathcal{B}(x) \\ \mathrm{diam}(R) < r}} \frac{1}{|R|} \int_R \chi_E.$$

Then $\mathcal{B}$ is a density basis.

Remark 1.21. The quantity

$$C_{\mathcal{B}}(\alpha) := \sup_{\substack{E \subset \mathbb{R}^n \\ 0 < |E| < \infty}} \frac{1}{|E|} |\{x \in \mathbb{R}^n : M_{\mathcal{B}} \chi_E(x) > \alpha\}|$$

is called the **Tauberian constant** associated to $\mathcal{B}$ and $\alpha \in (0, 1)$.

By the Hardy-Littlewood maximal theorem and the last result, the following holds.

Corollary 1.22. $\mathcal{Q}^*$ is a density basis.

In the example of the Hardy-Littlewood maximal operator, we saw that weak-type, and so differentiation properties of a basis, can be derived from covering result. In fact, this phenomenon is very general. It's the object of the next section.

1.3. **Covering properties**

The differentiation properties of a basis are deeply related to its covering properties. For example the Lebesgue differentiation theorem (theorem 1.8) is a consequence of a covering result for cubes. We are going to generalize this type of covering. Note that these results aren't usefull only in differentiation theory, but in many other fields. Firstly, we can generalize the idea developped for the Hardy-Littlewood maximal theorem. The following theorem is a generalization of lemma 1.17. There are many different versions of it, we refer to the first chapter of [10].

Theorem 1.23 (Besicovitch, 1945)**.** Let $A \subset \mathbb{R}^n$ be a bounded set. Assume

$$A \subset \bigcup_{x \in A} Q(x)$$

where $Q(x)$ is closed cubic interval centered at x. Then there exists a sequence of cubes $(Q_k)_k$ (which can be finite) such that :

(i)
$$A \subset \bigcup_k Q_k.$$

(ii) There exists a constant C such that for every $x \in \mathbb{R}^n$,
$$\sum_k \chi_{Q_k}(x) \leqslant C.$$

(iii) The sequence $(Q_k)_k$ can be decomposed into a finite collection of families of disjoint cubes.

Remark 1.24. The second point means that every point $x \in \mathbb{R}^n$ is in at most C balls B_k, this means that the balls B_k are of bounded overlap.

Definition 1.25. A basis $\mathcal{B}$ is called a Besicovitch basis if given a bounded set $A \subset \mathbb{R}^n$, and a set $R(x) \in \mathcal{B}(x)$ for each $x \in A$, one can choose from $(R(x))_{x \in A}$ a sequence $(R_k)_k$ such that A and $(R_k)_k$ satisfies the points (i), (ii) and (iii) of theorem 1.23.

Example 1.26. An example of another Besicovitch basis is given in [10, p118], in the plane. We consider the basis $\mathcal{B}$ invariant by homothecies generated by the rectangles $(R_k)_k$ such that for any k
$$\frac{L_k}{l_k} < 2,$$
where L_k (l_k) is the length of the longest (shortest) side of R_k. Then we can show that $\mathcal{B}$ is a Besicovitch basis, and then differentiates $L^1(\mathbb{R}^2)$.

Following the same proof as in theorem 1.18, we obtain the next result.

Theorem 1.27. Let $\mathcal{B}$ be a Besicovitch basis. The following holds.

- $M_{\mathcal{B}}$ is of weak-type $(1,1)$ and of strong-type (p,p) for $1 < p \leqslant \infty$.
- $\mathcal{B}$ differentiates $L^p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant \infty$.

If the open sets of our basis satisfy some inclusion hypothesis, theorem 1.27 applies (see [23, p24]) and we have the following.

Proposition 1.28. Let $\mathscr{F}$ be a one parametrer monotonic family of intervals in $\mathbb{R}^n$, *i.e.*

$$\mathscr{F} = \{R_t\}_{0<t<\infty}, \quad \text{and} \quad (s < t) \Rightarrow (R_s \subset R_t).$$

Then the basis
$$\mathcal{B} := \{x + R : x \in \mathbb{R}^n, R \in \mathscr{F}\}$$
differentiates $L^p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant \infty$.

We can also wonder in which way some differentiation properties imply corresponding covering results. This question is studied in chapter VII of [10]. We conclude this section by talking about some Vitali covering type, which is the most traditional tool to prove the Lebesgue differentiation theorem. The basics about it are given in the first chapter of [10]. In 1975, Córdoba showed that a Vitali covering property is in fact, under some light hypothesis, equivalent to differentiation properties. The results and the proofs can be found in [3]. We first introduce the notion of $\mathcal{B}$-Vitali covering and the property (V_q).

Definition 1.29. Let $\mathcal{B}$ be a Busemann-Feller basis, and $E \subset \mathbb{R}^n$ be measurable set. We say that $V \subset \mathcal{B}$ is a $\mathcal{B}$-Vitali covering of E if for every $x \in E$ there is a sequence $(R_k)_k \subset V$ such that $R_k \in \mathcal{B}(x)$ for each k and R_k shrinks to x as $k \to \infty$.

Definition 1.30. Let $1 \leqslant q < \infty$, and $\mathcal{B}$ be a Busemann-Feller basis. We say that $\mathcal{B}$ has the property (V_q) if for any measurable set E, every $\mathcal{B}$-Vitali covering V of E and any $\varepsilon > 0$, one can select a sequence $(R_k)_k \subset V$ such that

(i)

$$\left| E \backslash \bigcup_k R_k \right| = 0, \qquad \left| \bigcup_k R_k \backslash E \right| \leqslant \varepsilon,$$

(ii)

$$\left\| \sum_k \chi_{R_k} \right\|_q \lesssim |E|^{1/q}.$$

In his article, Córdoba has shown that the covering property (V_q) is in fact equivalent to differentiation properties.

Theorem 1.31 (Córdoba, 1975)**.** Let $\mathcal{B}$ be a Busemann-Feller basis invariant by translations. The following are equivalent.

(i) $\mathcal{B}$ differentiates $L^p(\mathbb{R}^n)$;

(ii) $\mathcal{B}$ has the covering property (V_q) ;

where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leqslant q < \infty$.

Remark 1.32. The special case $q = 1, p = \infty$ is due to de Possel.

1.4. **Sawyer-Stein principles**

Using basic tools of harmonic analysis, we basically saw that if $M_{\mathcal{B}}$ is of weak-type then $\mathcal{B}$ has good differentiation properties. What about the reverse ? Actually, the converse is almost true, it's the Sawyer-Stein principles. For $r > 0$ and $\mathcal{B}$ a basis, we introduce the truncated maximal operator

$$M_{\mathcal{B}}^r f(x) := \sup_{\substack{R \in \mathcal{B}(x) \\ \operatorname{diam}(R) \leqslant r}} \frac{1}{|R|} \int_R |f|.$$

To prove these Sawyer-Stein principles, we will need the following lemma (see [27, p165, II]).

Lemma 1.33. Let $(E_k)_{k\in\mathbb{N}^*}$ be a sequence of mesurable subsets of $\mathbb{R}^n$ such that

$$\sum_{k=1}^{\infty} |E_k| = \infty.$$

Then, there exists a collection $(\tau_k)_{k\in\mathbb{N}^*}$ of translation such that almost every point in $\mathbb{R}^n$ is contained in infinitely many of the sets $(\tau_k E_k)_{k\in\mathbb{N}^*}$.

1.4.1. **The cases, $L^p(\mathbb{R}^n)$, $p<\infty$, and $L^\Phi(\mathbb{R}^n)$**

Our goal is to prove the following.

Theorem 1.34 (Sawyer-Stein principle for $L^p(\mathbb{R}^n)$)**.** Let $\mathcal{B}$ be a Busemann-Feller basis invariant by translations. If $1 \leqslant p < \infty$, then the following are equivalent.

(i) There exists $r>0$ sucht that $M_{\mathcal{B}}^r$ is of weak-type (p,p).

(ii) $\mathcal{B}$ differentiates $L^p(\mathbb{R}^n)$.

The first step is to reduce the problem to a local problem.

Lemma 1.35. Let $\mathcal{B}$ be a Busemann-Feller basis invariant by translations. The following are equivalent.

(i) There exists $r>0$ sucht that $M_{\mathcal{B}}^r$ is of weak-type (p,p).

(i)' For each fixed cube Q, there exists $r>0$ (depending of Q) such that for all $f\in L^p(\mathbb{R}^n)$ with $\operatorname{supp}(f)\subset Q$ and each $\lambda>0$ we have

$$|\{x\in\mathbb{R}^n : M_{\mathcal{B}}^r f(x) > \lambda\}| \lesssim_Q \left(\frac{\|f\|_p}{\lambda}\right)^p.$$

Proof. Of course one has (i) $\Rightarrow$ (i)'. We prove the converse and suppose that (i)' is true. Then, for each cube Q two constants r_Q, C_Q are given. Since $\mathcal{B}$ is invariant by translations, it's clear that r_Q and C_Q do not depend on the place in $\mathbb{R}^n$ where Q is located, but only on its size. In addition, $M_{\mathcal{B}}^{r/2} \leqslant M_{\mathcal{B}}^r$ so we can assume that r_Q is less than half the length of the side of Q. We now fix a cube Q, and denote $r := r_Q$ and $C := C_Q$. Let $f\in L^1(\mathbb{R}^n)$, $f\geqslant 0$, with support contained in infinitely many disjoint cubic intervals $(Q_j)_{j\in\mathbb{N}^*}$ of the same length-size l. Then, as we have assumed, if $r\leqslant \frac{l}{2}$, for $\lambda>0$ one has

$$\{M_{\mathcal{B}}^r f > \lambda\} = \bigcup_{j=1}^{\infty} \{M_{\mathcal{B}}^r (f\chi_{Q_j}) > \lambda\}$$

and the sets $(\{M_{\mathcal{B}}^r(f\chi_{Q_j}) > \lambda\})_{j\in\mathbb{N}^*}$ are disjoint. Hence,

$$|\{M_{\mathcal{B}}^r f > \lambda\}| = \sum_{j=1}^{\infty} |\{M_{\mathcal{B}}^r (f\chi_{Q_j}) > \lambda\}| \leqslant \sum_{j=1}^{\infty} C\lambda^{-p}\int_{Q_j} f^p = C\left(\frac{\|f\|_p}{\lambda}\right)^p$$

(for each $j\geqslant 1$, $C_{Q_j} = C$ and $r_{Q_j} = r$). ☐

We can now prove the Sawyer-Stein principle.

Proof of the theorem. We already proved that (i) $\Rightarrow$ (ii). By the proposition 1.35, it suffices to show that (ii) $\Rightarrow$ (i)'. Assume (ii) and suppose that (i)' does not hold. Then, there exists a cube Q such that for each constants $c_k, r_k > 0$, there is a function $f_k \in L^p(\mathbb{R}^n)$ with $\operatorname{supp}(f_k) \subset Q$ such that

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}}^{r_k} f_k(x) > \lambda_k\}| \geqslant c_k \left(\frac{\|f_k\|_p}{\lambda_k}\right)^p.$$

for some $\lambda_k > 0$. We note $E_k := \{x \in \mathbb{R}^n : M_{\mathcal{B}}^{r_k} f_k(x) > \lambda_k\}$. Using lemma 1.33 and the sets E_k, we are going to construct a function that is not differentiable by $\mathcal{B}$. Let $(r_k)_k$ be sequence of positive numbers, $r_k \to 0$, such that all the numbers r_k are less than the side-length of Q, and let $c_k = 2^k$. We note $g_k = f_k/\lambda_k$. Let $\widetilde{Q}$ be the cube centered as Q but with 3 times his size. Clearly $E_k \subset \widetilde{Q}$ and

$$|E_k| = |\{x \in \mathbb{R}^n : M_{\mathcal{B}}^{r_k} g_k(x) > 1\}| \geqslant 2^k \|g_k\|_p^p. \tag{1}$$

For each $k \in \mathbb{N}^*$ we choose an integer m_k such that

$$|\widetilde{Q}| < m_k|E_k| < 2|\widetilde{Q}| \tag{2}$$

(it's possible because $|E_k| \leqslant \widetilde{Q}$). Then

$$\sum_{k=1}^{\infty} m_k|E_k| = \infty.$$

We define the sequence of sets $\{A_j\}_{j\in\mathbb{N}^*}$ by repeting m_k times the sets E_k, *i.e.*

$$\{A_1, \ldots, A_N, \ldots\} = \{\underbrace{E_1, \ldots, E_1}_{m_1 \text{ times}}, \ldots, \underbrace{E_N, \ldots, E_N}_{m_N \text{ times}}, \ldots\}.$$

Since

$$\sum_{j=1}^{\infty} |A_j| = \sum_{k=1}^{\infty} m_k|E_k| = \infty$$

we can apply the lemma 1.33. There exists some points

$$t_1^1, \ldots, t_1^{m_1}, \ldots, t_N^1, \ldots, t_N^{m_N}, \ldots$$

such that almost every point in $\mathbb{R}^n$ is containded in infinitely many of the sets $E_k^j + t_k^j$, where $E_k^j = E_k$ for $1 \leqslant j \leqslant m_k$. We define for all $k \geqslant 1$ the functions

$$g_k^j(x) := g_k(x - t_k^j), \quad 1 \leqslant j \leqslant m_k,$$

and

$$f_k(x) := \sup_{1\leqslant j\leqslant m_k} g_k^j(x), \quad f(x) := \sup_{k\in\mathbb{N}^*} f_k(x).$$

Then, for all $x \in \mathbb{R}^n$ we have $|f_k(x)|^p \leqslant \sum_{j=1}^{m_k} |g_k^j(x)|^p$, so

$$\|f_k\|_p^p \leqslant \sum_{j=1}^{m_k} \|g_k^j\|_p^p = m_k\|g_k\|_p^p.$$

We have to show that $f \in L^p(\mathbb{R}^n)$. With the inequalities (1) and (2), it comes

$$\|f\|_p^p \leqslant \sum_{k=1}^{\infty} \|f_k\|_p^p \leqslant \sum_{k=1}^{\infty} m_k\|g_k\|_p^p \leqslant \sum_{k=1}^{\infty} m_k|E_k|2^{-k} \leqslant 2|\widetilde{Q}|\sum_{k=1}^{\infty} 2^{-k} < \infty.$$

Now, let's show that the integral of f isn't differentiable by $\mathcal{B}$. Almost every $x \in \mathbb{R}^n$ belongs to an infinite number of sets of the form $E_k^j + t_k^j$. Then, for almost every $x \in \mathbb{R}^n$ there exist two strictly increasing sequences $\{k_x\}, \{j_x\} \subset \mathbb{N}^*$ such that

$$x \in E_{k_x}^{j_x} + t_{k_x}^{j_x}.$$

By the definition of the sets E_k^j, for $x \in E_{k_x}^{j_x} + t_{k_x}^{j_x}$ we have

$$M_{\mathcal{B}}^{r_{k_x}} f(x) \geqslant M_{\mathcal{B}}^{r_{k_x}} g_{k_x}^{j_x}(x) > 1$$

Then for almost every $x \in \mathbb{R}^n$ there exists a sequence $\{R_k(x)\}_{k \in \mathbb{N}^*} \subset \mathcal{B}(x)$, of diameters tending to 0 such that

$$\frac{1}{|R_k(x)|} \int_{R_k(x)} f > 1.$$

We supposed that $\mathcal{B}$ differentiates $L^p(\mathbb{R}^n)$, since $f \in L^p(\mathbb{R}^n)$ we know that

$$\lim_{k \to \infty} \frac{1}{|R_k(x)|} \int_{R_k(x)} f = f(x)$$

for almost every $x \in \mathbb{R}^n$, and so we have $f \geqslant 1$ almost everywhere. We obtain a contradiction because f has to be in $L^p(\mathbb{R}^n)$. □

Remark 1.36. If we further assume that the basis is homothecy invariant, then we can replace $M_{\mathcal{B}}^r$ by $M_{\mathcal{B}}$ in theorem 1.34, see [10, Theorem 3.4].

In fact, following exactly the same proof, the Sawyer-Stein principle remains valid in Orlicz spaces. Indeed, during the proof we never used any particular property of the space $L^p(\mathbb{R}^n)$, only the weak-type inequality matters.

Theorem 1.37 (Sawyer-Stein principle for $L^\Phi(\mathbb{R}^n)$)**.** Let $L^\Phi(\mathbb{R}^n)$ be an Orlicz space, and $\mathcal{B}$ a Busemann-Feller basis invariant by translations. Then, the following are equivalent.

(i) There exists $r > 0$ such that $M_{\mathcal{B}}^r$ is of weak-type (Φ, Φ).

(ii) $\mathcal{B}$ differentiates $L^\Phi(\mathbb{R}^n)$.

1.4.2. The case $L^\infty(\mathbb{R}^n)$

We also have a version of the Sawyer-Stein principle for the density bases. We refer to [14] for the proof, there are a few similarities with the $L^p(\mathbb{R}^n)$ case.

Theorem 1.38 (Sawyer-Stein principle for $L^\infty(\mathbb{R}^n)$)**.** Let $\mathcal{B}$ be Busemann-Feller basis invariant by translations. The following are equivalent.

(i) $\mathcal{B}$ is a density basis.

(ii) For any $\alpha \in (0,1)$, there exists a real $r > 0$ and a constant $C < \infty$ such that for each measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$ we have

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}}^r \chi_E(x) > \alpha\}| \leqslant C(\alpha)|E|.$$

Remark 1.39. If we further assume that the basis is homothecy invariant, then we can replace $M_{\mathcal{B}}^r$ by $M_{\mathcal{B}}$ in the theorem, see [10, Theorem 1.2 p69].

Theorem 1.38 is very useful for constructing counterexamples. For example, there are bases that are not density bases. Let's construct one of them. We consider the basis $\mathcal{B}$ defined by

$$\mathcal{B} := \left\{x + \Gamma(a,b,c,d) : a,b,c,d > 0, x \in \mathbb{R}^2\right\}$$

where $\Gamma(a,b,c,d)$ is the open set defined below in the figure 3. Let $Q = (0,1)^2$ be the unit cube of $\mathbb{R}^2$, and $A := \{(x,y) \in \mathbb{R}^2 : 0 < y < 1\}$. If $(x,y) \in A$, it's clear that we can find a set $R \in \mathcal{B}$ containing (x,y) such that $\frac{|R \cap Q|}{|R|} > \frac{1}{2}$, so $A \subset \left\{M_{\mathcal{B}}\chi_Q > \frac{1}{2}\right\}$ and

$$\left|\left\{M_{\mathcal{B}}\chi_Q > \frac{1}{2}\right\}\right| = \infty.$$

By theorem 1.38, $\mathcal{B}$ is not a density basis.

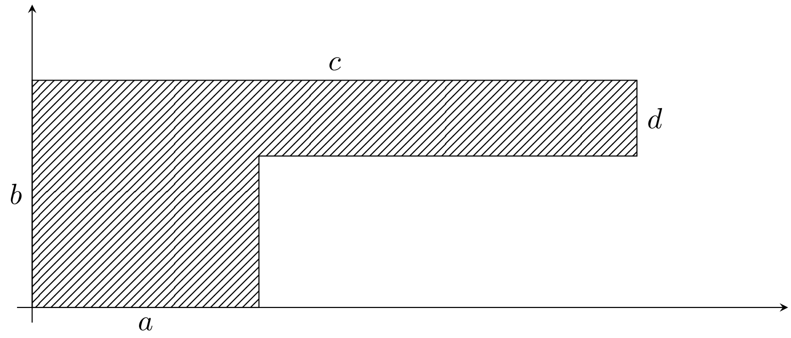

Figure 3: The set Γ.

2. Rectangles with sides parallel to coordinate axes

2.1. The strong maximal operator

In this section we take $n \geqslant 2$ (otherwise the strong maximal operator is just the unidimensional Hardy-Littlwood maximal operator). We have seen that $\mathcal{Q}$ differentiates $L^1(\mathbb{R}^n)$, this is Lebesgue's derivation theorem. In this section, we study the differentiation basis $\mathcal{I}$. Since the differentiation basis $\mathcal{I}$ is a Busemann-Feller basis, the measurability of M_{s} is ensured by proposition 1.14. With the help of theorem 1.18, we have the following.

Proposition 2.1. Let $1 < p \leqslant \infty$. Then the strong maximal operator M_{s} is of strong-type (p, p).

Proof. The case $p = \infty$ holds for each maximal operator. Assume that $1 < p < \infty$. We denote M_{s}^* the maximal operator associated to $\mathcal{I}^*$. As in example 1.13, there exists a constant C_n such that, everywhere on $\mathbb{R}^n$ one has

$$M_{\mathrm{s}}^* f \leqslant M_{\mathrm{s}} f \leqslant C_n M_{\mathrm{s}}^* f$$

for each $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Thus it's enough to show that M_{s}^* is of strong type (p, p). Let $f \in L^p(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
M_{\mathrm{s}}^* f(x) &= \sup_{r_1,\ldots,r_n>0} \frac{1}{|\prod_{i=1}^n (-r_i, r_i)|} \int_{\prod_{i=1}^n (-r_i,r_i)} |f(x-y)| \mathrm{d}y \\
&\leqslant \sup_{r_1,\ldots,r_{n-1}>0} \frac{1}{|\prod_{i=1}^{n-1} (-r_i, r_i)|} \int_{\prod_{i=1}^{n-1} (-r_i,r_i)} \frac{1}{|(-r_n, r_n)|} \sup_{r_n>0} \int_{(-r_n,r_n])} |f(x'-y', x_n - y_n)| \mathrm{d}y' \mathrm{d}y_n \\
&\leqslant \sup_{r_1,\ldots,r_{n-1}>0} \frac{1}{|\prod_{i=1}^{n-1} (-r_i, r_i)|} \int_{\prod_{i=1}^{n-1} (-r_i,r_i)} \frac{1}{|(-r_n, r_n)|} M_{\mathrm{HL}}^{(n)} f(x'-y', x_n) \mathrm{d}y',
\end{aligned}$$

where $M_{\mathrm{HL}}^{(n)}$ is the Hardy-Littlewood maximal operator in the nth variable only, and $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. By repeating this computation, the Fubini's theorem and the theorem 1.18 leads to the desired result. □

It's natural to wonder what happens for $p = 1$. We can expect to obtain a weak-type $(1, 1)$, like the Hardy-Littlewood maximal operator, but in fact not at all.

Proposition 2.2. The strong maximal operator M_{s} is not of weak-type $(1, 1)$.

Proof. Firstly, and in fact the main work is here, we will focus on the case $n = 2$. We begin by exhibiting a useful construction in the plane. Let $N \geqslant 1$ be an integer. Let $i \in \{1, \ldots, N\}$. Consider the rectangle $R_i \in \mathcal{I}$, whose bottom left-hand corner has coordinates $(0, 0)$ and whose top right-hand corner has coordinates $(i, N/i)$ (see figure 2.1). Then we define

$$I_N := \bigcap_{i=1}^N R_i, \quad U_N := \bigcup_{i=1}^N R_i.$$

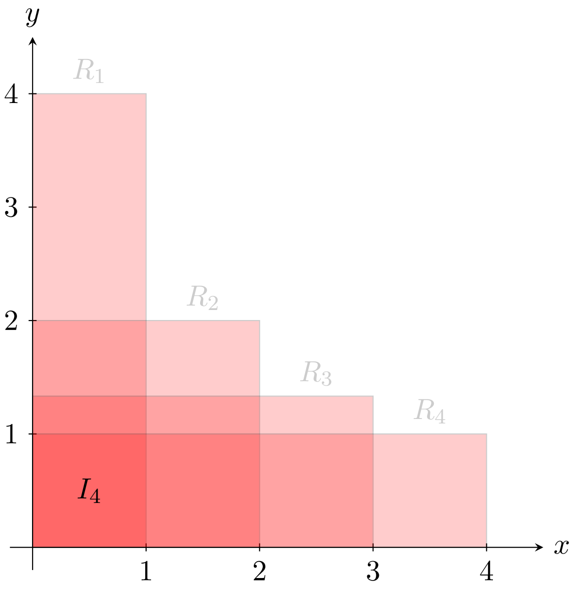

Figure 4: The construction for $N = 4$.

The area of I_N is therefore 1, while

$$|U_N| = N\left(1 + \frac{1}{2} + \cdots \frac{1}{N}\right) \underset{N\to\infty}{\sim} N\ln(N).$$

Using this construction, let's show that M_{s} cannot verify a weak-type inequality $(1,1)$ for the indicator function of I_N. Let $x \in U_N$. Then there exists $j \in [\![1, N]\!]$ such that $x \in R_j$. Hence,

$$M_{\mathrm{s}}(\chi_{I_N})(x) \geqslant \frac{1}{|R_j|}\int_{R_j}\chi_{I_N} = \frac{|I_N \cap R_j|}{|R_j|} = \frac{1}{N} > \frac{1}{2N}.$$

Therefore,

$$U_N \subset \left\{M_{\mathrm{s}}(\chi_{I_N}) > \frac{1}{2N}\right\}$$

and

$$\left|\left\{M_{\mathrm{s}}(\chi_{I_N}) > \frac{1}{2N}\right\}\right| \geqslant |I_N| N\left(1 + \frac{1}{2} + \cdots \frac{1}{N}\right)$$

This contradicts the weak-type $(1,1)$ inequality for N large enough.
For the general case $n \geqslant 2$, it suffices to consider the sets

$$\widetilde{I}_N := I_N \times [0,1]^{n-2}, \quad \widetilde{U}_N := U_N \times [0,1]^{n-2}.$$

We conclude the proof in a similar way. □

By the Sawyer-Stein principle, we know that there are integrable functions for which the integral are not differentiated by $\mathcal{I}$, that is to say that $\mathcal{I}$ does not differentiate $L^1(\mathbb{R}^n)$. On the other hand, $\mathcal{I}$ differentiates $L^2(\mathbb{R}^n)$ (proposition 2.1). The question is : is there an intermediate space between $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ such that $\mathcal{I}$ differentiates (precisely) this space? The answer is yes : it's the Orlicz space $L^\Phi(\mathbb{R}^n)$ for $\Phi(t) := t(1 + \log^+ t)^{n-1}$.

Theorem 2.3 (Jessen, Marcinkiewicz, Zygmund, 1935). Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then

$$|\{x \in \mathbb{R}^n : M_{\mathrm{s}}f(x) > \lambda\}| \lesssim_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\frac{|f(x)|}{\lambda}\right)^{n-1} \mathrm{d}x,$$

for each $\lambda > 0$.

Remark 2.4. For $\Phi(t) = t(1+\log^+ t)^k$, the space $L^\Phi(\mathbb{R}^n)$ will be denoted $L(\log^+ L)^k(\mathbb{R}^n)$. Indeed, if $\widetilde{\Phi}(t) := t(\log^+ t)^k$, then locally $L^\Phi(\mathbb{R}^n) = L^{\widetilde{\Phi}}(\mathbb{R}^n)$.

The analogue of Lebesgue's differentiation theorem,

$$f(x) = \lim_{\substack{\operatorname{diam}(R)\to 0 \\ R\in\mathcal{I}(x)}} \frac{1}{|R|}\int_R f(y)\mathrm{d}y, \quad \text{for almost all } x \in \mathbb{R}^n$$

is therefore false for some functions $f \in L^1(\mathbb{R}^n)$, but remains true if $f \in L\log^+ L(\mathbb{R}^n)$. By theorem 1.38, we know that $\mathcal{I}$ is also a density basis. We'll prove the theorem in the plane case, $n = 2$. In the next section, we'll look at a way of generalizing it. We take the proof from de Guzmán's book, see [10].

Proof. Let $f \in L^1_{\text{loc}}(\mathbb{R}^2)$, $f \geqslant 0$. The idea is to use Hardy-Littlewood's maximal theorem, as well as Fubini's theorem, in a clever way. For $(x_1, x_2) \in \mathbb{R}^2$ and $\lambda > 0$, we define

$$T_1 f(x_1, x_2) := \sup\left\{\frac{1}{|J|}\int_J f(\xi_1, x_2)\mathrm{d}\xi_1 : J \text{ interval of } \mathbb{R} \text{ containing } x_1\right\};$$

$$A := \left\{(\eta_1, \eta_2) \in \mathbb{R}^2 : T_1 f(\eta_1, \eta_2) > \lambda/2\right\};$$

and

$$T_2 f(x_1, x_2) := \sup\left\{\frac{1}{|H|}\int_H \chi_A(x_1, \eta_2) T_1 f(x_1, \eta_2)\mathrm{d}\eta_2 : H \text{ interval of } \mathbb{R} \text{ containing } x_2.\right\}.$$

We'll show that if $M_{\mathrm{s}} f$ is large at a point, then $T_2 f$ is also large at that point. Let's show that

$$B := \{(\xi_1, \xi_2) \in \mathbb{R}^2 : M_{\mathrm{s}} f(\xi_1, \xi_2) > \lambda\} \subset \{(\xi_1, \xi_2) \in \mathbb{R}^2 : T_2 f(\xi_1, \xi_2) > \lambda/2\} =: C.$$

Let $(x_1, x_2) \in \mathbb{R}^2$ be such that $M_{\mathrm{s}} f(x_1, x_2) > \lambda$. So there is a rectangle $R = J \times H$ of $\mathbb{R}^2$ such that

$$x_1 \in J, \quad x_2 \in H, \quad \frac{1}{|R|}\int_R f > \lambda.$$

Let's cut out the rectangle R as follows. We write $R = C_1 \cup C_2$, where :

- if for all $(z_1, y_2) \in J \times \{y_2\}$, we have $T_1 f(z_1, y_2) > \lambda/2$, then $J \times \{y_2\} \subset C_1$;
- otherwise, if there exists a point $(z_1, y_2) \in J \times \{y_2\}$ such that $T_1 f(z_1, y_2) \leqslant \lambda/2$, then $J \times \{y_2\} \subset C_2$.

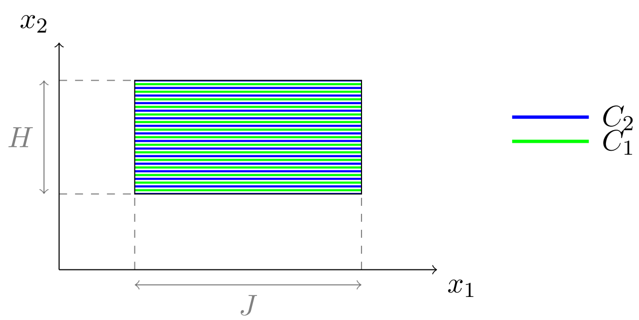

Figure 5: $R = C_1 \cup C_2$

Note that

$$J \times \{y_2\} \subset C_2 \Rightarrow \frac{1}{|J|}\int_J f(\xi_1, y_2)\mathrm{d}\xi_1 \leqslant \frac{\lambda}{2}.$$

So we'll integrate this inequality on the set $\{\xi_2 \in H : J \times \{\xi_2\} \subset C_2\} =: D$, then we get

$$\int_{C_2} f = \int_D \int_J f(\xi_1, \xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \leqslant \int_D \frac{\lambda}{2} |J| \mathrm{d}\xi_2 = \frac{\lambda}{2}|C_2| \leqslant \frac{\lambda}{2}|I|,$$

because $|C_2| = |J||D|$ and $|R| = |C_1| + |C_2|$. One has

$$\int_{C_1} f + \int_{C_2} f = \int_R f > \lambda |I|,$$

it comes

$$\int_{C_1} f > \frac{\lambda}{2}|R|. \tag{$\star$}$$

We also have

$$\begin{aligned} T_2 f(x_1, x_2) &\geqslant \frac{1}{|H|} \int_H \chi_A(x_1, \eta_2) T_1 f(x_1, \eta_2) \mathrm{d}\eta_2 \\ &\geqslant \frac{1}{|H|} \int_H \chi_A(x_1, \eta_2) \frac{1}{|J|} f(\eta_1, \eta_2) \mathrm{d}\eta_1 \mathrm{d}\eta_2 \\ &= \frac{1}{|R|} \int_H \chi_A(x_1, \eta_2) f(\eta_1, \eta_2) \mathrm{d}\eta_1 \mathrm{d}\eta_2. \end{aligned}$$

But $C_1 \subset A$. Indeed, by definition of C_1

$$(\eta_1, \eta_2) \in C_1 \Rightarrow (x_1, \eta_2) \in C_1 \Rightarrow T_1 f(x_1, \eta_2) > \frac{\lambda}{2} \Rightarrow (x_1, \eta_2) \in A.$$

Using inequality ($\star$), we finally have

$$T_2 f(x_1, x_2) \geqslant \frac{1}{|R|} \int_{C_1} f(\eta_1, \eta_2) \mathrm{d}\eta_1 \mathrm{d}\eta_2 \geqslant \frac{\lambda}{2}.$$

Hence we have the inclusion $B \subset C$. We now assume that $f \in L^1(1 + \log^+ L^1)(\mathbb{R}^n)$ (the result being trivial otherwise). By Hardy-Littlewood's maximal theorem, we have

$$|\{\xi_2 \in \mathbb{R} : T_2 f(x_1, \xi_2) > \lambda/2\}| \lesssim \frac{2}{\lambda} \|T_2 f(x_1, \cdot)\|_{L^1(\mathbb{R}^n)} = \frac{2}{\lambda} \int_{\mathbb{R}} \chi_A(x_1, \xi_2) T_1 f(x_1, \xi_2) \mathrm{d}\xi_2.$$

We integrate over the set of $x_1 \in \mathbb{R}$ then change the order of integration :

$$|C| = |\{(\xi_1, \xi_2) \in \mathbb{R}^2 : T_2 f(\xi_1, \xi_2) > \lambda/2\}| \lesssim \frac{2}{\lambda} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_A(\xi_1, \xi_2) T_1 f(\xi_1, \xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2.$$

We know that if $g \in L^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} |g(x)| \mathrm{d}x = \int_0^\infty |\{t \in \mathbb{R} : |g(t)| > \lambda\}| \mathrm{d}\lambda.$$

Here, we get

$$|C| \lesssim \int_{\mathbb{R}} \int_0^\infty \left| \left\{ \xi_1 \in \mathbb{R} : \frac{\chi_A(\xi_1, \xi_2) T_1 f(\xi_1, \xi_2)}{\lambda/2} > \sigma \right\} \right| \mathrm{d}\sigma \mathrm{d}\xi_2 = I_1 + I_2$$

where

$$I_1 = \int_{\mathbb{R}} \int_0^1 \left| \left\{ \xi_1 \in \mathbb{R} : \frac{\chi_A(\xi_1, \xi_2) T_1 f(\xi_1, \xi_2)}{\lambda/2} > \sigma \right\} \right| \mathrm{d}\sigma \mathrm{d}\xi_2$$

and

$$I_2 = \int_{\mathbb{R}} \int_1^\infty \left| \left\{ \xi_1 \in \mathbb{R} : \frac{\chi_A(\xi_1, \xi_2) T_1 f(\xi_1, \xi_2)}{\lambda/2} > \sigma \right\} \right| \mathrm{d}\sigma \mathrm{d}\xi_2.$$

We'll estimate these integrals differently.

- Fix any $0 < \sigma \leqslant 1$. If $(\xi_1, \xi_2) \in A$ then $T_1 f(\xi_1, \xi_2) > \lambda/2$. So :

$$\left| \left\{ \xi_1 \in \mathbb{R} : \frac{\chi_A(\xi_1, \xi_2) T_1 f(\xi_1, \xi_2)}{\lambda/2} > \sigma \right\} \right| = \left| \{ \xi_1 \in \mathbb{R} : T_1 f(\xi_1, \xi_2) > \lambda/2 \} \right|.$$

  Once again, we use the Hardy-Littlewood theorem,

$$I_1 \lesssim \int_{\mathbb{R}} \left| \{ \xi_1 \in \mathbb{R} : T_1 f(\xi_1, \xi_2) > \lambda/2 \} \right| \mathrm{d}\xi_2 \lesssim \frac{2}{\lambda} \int_{\mathbb{R}^2} f(\xi_1, \xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2.$$

- To estimate I_2, we truncate f according to its large values. We define for a fixed $\sigma > 0$

$$f_{\sigma,*}(\xi_1, \xi_2, \sigma) := \begin{cases} f(\xi_1, \xi_2) \text{ if } f(\xi_1, \xi_2) \leqslant \frac{\lambda\sigma}{4}, \\ 0 \text{ if } f(\xi_1, \xi_2) > \frac{\lambda\sigma}{4}, \end{cases}$$

  and $f_\sigma^* := f - f_{\sigma,*}$. Since $T_1 f \leqslant T_1 f_\sigma^* + T_1 f_{\sigma,*}$ and $T_1 f_{\sigma,*} \leqslant \frac{\lambda\sigma}{4}$, we have

$$\left| \left\{ \xi_1 \in \mathbb{R} : \chi_A T_1 f(\xi_1, \xi_2) > \frac{\lambda\sigma}{2} \right\} \right| \leqslant \left| \left\{ \xi_1 \in \mathbb{R} : T_1 f_\sigma^*(\xi_1, \xi_2) > \frac{\lambda\sigma}{4} \right\} \right|.$$

  Hence,

$$\begin{aligned} I_2 &\lesssim \int_{\mathbb{R}} \int_1^\infty \left| \left\{ \xi_1 \in \mathbb{R} : T_1 f_\sigma^*(\xi_1, \xi_2) > \frac{\lambda\sigma}{4} \right\} \right| \mathrm{d}\sigma \mathrm{d}\xi_2 \\ &\lesssim \int_{\mathbb{R}} \int_1^\infty \int_{\mathbb{R}} \frac{4}{\lambda\sigma} f_\sigma^*(\xi_1, \xi_2) \mathrm{d}\xi_1 \mathrm{d}\sigma \mathrm{d}\xi_2 \\ &\lesssim \int_{\mathbb{R}^2} \int_1^{\frac{4f(\xi_1,\xi_2)}{\lambda}} \frac{f_\sigma^*(\xi_1, \xi_2)}{\lambda\sigma} \mathrm{d}\sigma \mathrm{d}\xi_1 \mathrm{d}\xi_2, \end{aligned}$$

  where we used the Hardy-Littlewood's theorem and Fubini's theorem. Next, a calculation leads to

$$I_2 \lesssim \int_{\mathbb{R}^2} \frac{f(\xi_1, \xi_2)}{\lambda} \log^+ \frac{4f(\xi_1, \xi_2)}{\lambda} \mathrm{d}\xi_1 \mathrm{d}\xi_2.$$

By putting together the estimates of I_1 and I_2, we can conclude the proof. □

It is possible to establish another proof of this theorem, based on geometric arguments. This proof is due to Córdoba and Fefferman, see [5], and it's based on theorem 1.31. This proof is interesting because it gives us a better understanding of the geometry of rectangles. Furthermore, we know that the inequality of the theorem 2.3 is the best that we can hope for. Indeed, this weak-type inequality implies that $\mathcal{I}$ differentiates $L\log^+ L(\mathbb{R}^2)$, but in 1935, Saks proved that it's the best space differentiable by $\mathcal{I}$, see [20].

Theorem 2.5 (Saks, 1935)**.** The basis $\mathcal{I}$ does not differentiate $o(L\log^+ L(\mathbb{R}^2))$.

We have the immediate corollary.

Corollary 2.6. The basis $\mathcal{I}$ differentiates precisely $L\log^+ L(\mathbb{R}^2)$.

2.2. Restriction on the number of sides with different length

Up to now, we deal with the basis $\mathcal{Q}$ and the basis $\mathcal{I}$ (in the plane). The first one differentiates $L^1(\mathbb{R}^n)$, and the second one differentiates (precisely) $L(\log^+ L)^{n-1}(\mathbb{R}^n)$ (we proved it for $n = 2$, but the result holds for each $n \geqslant 1$, as we'll soon see). The goal of this section is to have a first look on an intermediate situation, *i.e.* what happens for a basis $\mathcal{B}$ such that

$$\mathcal{Q} \subset \mathcal{B} \subset \mathcal{I}.$$

We still consider rectangles with sides parallel to axes, the restriction on $\mathcal{B}$ are going to concern the number of sides with different length.

Definition 2.7. For $1 \leqslant s \leqslant n$, we note $\mathcal{B}_s \subset \mathcal{I}_n$ the basis such that for all $x \in \mathbb{R}^n$, $\mathcal{B}_s(x)$ is the collection of open bounded intervals containing x such that s of the n side length are equal.

Remark 2.8. Of course, $\mathcal{B}_1 = \mathcal{I}$ and $\mathcal{B}_n = \mathcal{Q}$.

The theory about the bases $\mathcal{B}_s$ is based on the strong maximal theorem. We will need the following result, see [9] for more details.

Proposition 2.9. Let $\mathcal{B}_i$, $i = 1, 2$ be two bases in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$. We assume that $M_{\mathcal{B}_i}$ is of weak-type (Φ_i, Φ_i), for $i = 1, 2$, where Φ_i are Young functions. We note $\mathcal{B} \subset \mathbb{R}^{n_1+n_2}$ the product basis of $\mathcal{B}_1$ and $\mathcal{B}_2$, *i.e.*

$$\mathcal{B} = \{R_1 \times R_2 : (R_1, R_2) \in \mathcal{B}_1 \times \mathcal{B}_2\}.$$

Then, for each $\lambda > 0$ and $f \in L^1_{\text{loc}}(\mathbb{R}^{n_1+n_2})$

$$\begin{aligned} &\left|\left\{x \in \mathbb{R}^{n_1+n_2} : M_{\mathcal{B}} f(x) > \lambda\right\}\right| \\ &\leqslant \Phi_2(1) \int_{\mathbb{R}^{n_1+n_2}} \Phi_1\left(\frac{2|f|}{\lambda}\right) \mathrm{d}x + \int_{\mathbb{R}^{n_1+n_2}} \left[\int_1^{\frac{4|f(x)|}{\lambda}} \Phi_1\left(\frac{4|f(x)|}{\lambda\sigma}\right) \mathrm{d}\Phi_2(\sigma)\right] \mathrm{d}x. \end{aligned}$$

By applying several times proposition 2.9, we have the following generelization obtained by Zygmund in 1967, see [28].

Theorem 2.10 (Zygmund, 1967)**.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $1 \leqslant s \leqslant n$. Then

$$|\{x \in \mathbb{R}^n : M_{\mathcal{B}_s} f(x) > \lambda\}| \lesssim_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda}\right)^{n-s} \mathrm{d}x,$$

pour tout $\lambda > 0$.

As in the case of $\mathcal{I}$, the space $L(\log^+ L)^{n-s}(\mathbb{R}^n)$ is in fact the best differentiable space by $\mathcal{B}_s$, and therefore the last inequality is the best we can have. We can refer to [24, Theorem 2] for the proof.

Theorem 2.11. The basis $\mathcal{B}_s$ does not differentiates $o(L(\log^+ L)^{n-s}(\mathbb{R}^n))$.

Corollary 2.12. The basis $\mathcal{B}_s$ differentiates precisely $L(\log^+ L)^{n-s}(\mathbb{R}^n)$.

Remark 2.13. By applying the theorem for $s \in \{1, n\}$, we obtain the theorems 2.3 and 1.18.

It's interesting to note that locally, we have the inclusions

$$L^1(\mathbb{R}^n) \supset \cdots \supset L(\log^+ L)^{n-s}(\mathbb{R}^n) \supset \cdots \supset L(\log^+ L)^{n-1}(\mathbb{R}^n),$$

and at the same time we obviously have

$$\mathcal{Q} = \mathcal{B}_n \subset \cdots \subset \mathcal{B}_s \subset \cdots \subset \mathcal{B}_1 = \mathcal{I}.$$

We seem to notice that if a basis has few elements, then it will have good differentiation properties. But what is the meaning of "few elements" ? The next section will answer that, at least for the case $n = 2$.

2.3. A geometric characterization for rectangles in the plane

We focus on the case $n = 2$. We saw that $\mathcal{Q}$ differentiates $L^1(\mathbb{R}^2)$, and $\mathcal{I}$ differentiates $L\log^+ L(\mathbb{R}^2)$. Given a basis $\mathcal{B}$ invariant by translations such that $\mathcal{Q} \subset \mathcal{B} \subset \mathcal{I}$, it's clear that $\mathcal{B}$ differentiates $L\log^+ L(\mathbb{R}^2)$. But does $\mathcal{B}$ differentiate $L^1(\mathbb{R}^2)$, or any other intermediate space between those two ? In 1988, in [25], Stokolos proved that there are only two possibilities : either $\mathcal{B}$ differentiates $L^1(\mathbb{R}^n)$, or $\mathcal{B}$ does not differentiates $o(L\log^+ L(\mathbb{R}^2))$. Furthermore, Stokolos has given us a geometric criterion to determine which case we are.

Definition 2.14. Let $R, R' \subset \mathcal{I}$ be two rectangles. R and R' are said to be **comparable**, and we note $R \sim R'$, if we can include one in the other via a translation. Otherwise, they are said to be **incomparable**, and we note $R \not\sim R'$.

Remark 2.15. The relation $\sim$ is not an equivalent relation, transitivity fails (see figure 7).

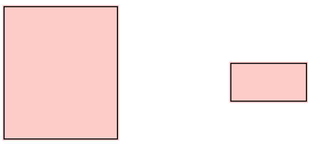

Figure 6: Comparable rectangles.

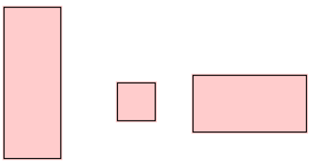

Figure 7: Incomparable rectangles.

Like usually in harmonic analysis, we'll work with dyadic decomposition, and here with dyadic cubes. If $R \in \mathcal{I}$, we define $R^* \in \mathcal{I}$ as the cocentric rectangle of minimal measure containing R with side-lengths of the side 2^k, for $k \in \mathbb{Z}$. If $\mathcal{B}(x) \subset \mathcal{I}_2$ is a basis fo $x \in \mathbb{R}^2$, we denote

$$\mathcal{B}^*(x) = \{R^* : R \in \mathcal{B}(x)\}$$

and $\mathcal{B}^* : x \in \mathbb{R}^2 \longmapsto \mathcal{B}(x)$.

Definition 2.16. Let $\mathcal{B}$ be a basis. We'll say that $\mathcal{B}$ has the property (S) if

$$\forall \varepsilon > 0, \forall k \in \mathrm{N}, \exists \{R_i\}_{i=1}^k \subset \mathcal{B}^* : \quad \begin{array}{l} \forall i \neq j, R_i \not\sim R_j, \\ \forall i \in [\![1, k]\!], \mathrm{diam}(R_i) < \varepsilon. \end{array}$$

In other words, $\mathcal{B}$ has (S) if we can find an arbitrarily number of arbitrarily small pairwise incomparable rectangles in the associated basis.

Theorem 2.17 (Stokolos, 1988)**.** Let $B \subset \mathcal{I}$ be a translation invariant basis. Then :

- if $\mathcal{B}$ has property (S), then $\mathcal{B}$ does not differentiates $o(L\log^+ L(\mathbb{R}^2))$;
- if $\mathcal{B}$ fails to have property (S), then $\mathcal{B}$ differentiates $L^1(\mathbb{R}^2)$.

Remark 2.18. Since $\mathcal{B} \subset \mathcal{I}$, if $\mathcal{B}$ does not differentiates $o(L\log^+ L(\mathbb{R}^2))$ then $\mathcal{B}$ differentiates precisely $L\log^+ L(\mathbb{R}^2)$.

We will need the following lemmas.

Lemma 2.19. Let $(a_k)_{k\geqslant 1}$, $(b_k)_{k\geqslant 1}$ be two sequences of non-negative numbers. We assume that $\sum_{k=1}^{\infty} a_k < \infty$ and $b_k \longrightarrow 0$. Then, there exists a sequence of integers $(m_k)_{k\geqslant 1} \subset \mathbb{N}^*$ such that

$$\sum_{k=1}^{\infty} a_k m_k = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} a_k b_k m_k < \infty.$$

Proof. Let $(k_l)_{l\geqslant 1}$ be a strictly increasing sequence such that for all $l \in \mathbb{N}^*$ we have

$$b_{k_l} \leqslant \frac{1}{l^2}.$$

For $l \in \mathbb{N}^*$, we choose $\widetilde{m_l} \in \mathbb{N}^*$ an integer such that

$$\frac{1}{l^3 a_{k_l} b_{k_l}} \leqslant \widetilde{m_l} \leqslant 2 + \frac{1}{l^3 a_{k_l} b_{k_l}}.$$

Then it comes that

$$\sum_{l=1}^{\infty} a_{k_l}\widetilde{m_l} = \infty \quad \text{and} \quad \sum_{l=1}^{\infty} a_{k_l} b_{k_l}\widetilde{m_l} < \infty.$$

To obtain the correct form of the lemma, we can define the sequence $(m_k)_{k\geqslant 1}$ as

$$m_k = \begin{cases} \widetilde{m_l} & \text{if } k \in \{k_j\} \text{ and } k = k_l, \\ 1 & \text{otherwise.} \end{cases}$$

□

The second lemma is the main tool in the proof of the theorem, it's a covering property. See [25, Lemma 1] for the proof.

Lemma 2.20. Let $\mathcal{B}$ be a basis with the property (S). Then, for any $\varepsilon > 0$ and $k \in \mathbb{N}$, there are sets Θ and Y such that

$$\Theta \subset Y, \quad \operatorname{diam}(Y) < \varepsilon, \quad |Y| \geqslant k2^{k-1}|\Theta|,$$

and

$$\forall\, x \in Y, \exists\, R \in \mathcal{B}^*(x) : \quad \operatorname{diam}(R) < \varepsilon, \quad \frac{|R \cap \Theta|}{|R|} \geqslant 2^{-k}.$$

We can now prove the theorem.

Proof of the theorem. We suppose that $\mathcal{B}$ has property (S), and let g be as in the definition 1.9. We note $\Phi : t \longmapsto g(t)t\log^+ t$. Denote by $\widetilde{\mathcal{B}}$ the basis obtained from $\mathcal{B}$ by dilation with coefficient

$\frac{1}{2}$. Applying the lemma 2.20 with $\varepsilon = \varepsilon_k \underset{k\to\infty}{\longrightarrow} 0$ and $k \in \mathbb{N}$, we obtain sequences of sets $(\Theta_k)_{k\geqslant 1}$, $(Y_k)_{k\geqslant 1}$ satisfying the lemma's conclusions. First of all, we have

$$\operatorname{diam}(Y_k) \leqslant \varepsilon_k \underset{k\to\infty}{\longrightarrow} 0.$$

Since for each $y \in Y_k$, Y_k is included in the ball of center y and of radius $\operatorname{diam}(Y_k)$, we have $|Y_k| \lesssim_n |\operatorname{diam}(Y)|^n \lesssim_n \varepsilon_k^n$. For the sets Θ_k, it implies that

$$|\Theta_k| \lesssim_n \frac{\varepsilon_k^n}{k2^{k-1}},$$

so by choosing $(\varepsilon_k)_k$ small enough, we can assume that

$$\sum_{k=1}^{\infty} k2^k|\Theta_k| < \infty$$

(for example $\varepsilon_k = 2^{-k/n}$). Furthermore, by the lemma 2.19 there exists some numbers $(m_k)_{k\geqslant 1}$ such that

$$\sum_{k=1}^{\infty} k2^k|\Theta_k|m_k = \infty, \quad \sum_{k=1}^{\infty} g(2^k)k2^k|\Theta_k|m_k = \frac{1}{\log 2}\sum_{k=1}^{\infty}\Phi(2^k)|\Theta_k|m_k < \infty. \tag{3}$$

Moreover, let α_k be numbers such that $\alpha_k \underset{k\to\infty}{\longrightarrow} \infty$ and

$$\sum_{k=1}^{\infty} g(\alpha_k 2^k)k\alpha_k 2^k|\Theta_k|m_k < \infty. \tag{4}$$

Let's $(N_j)_{j\in\mathbb{N}}$ be defined by

$$N_0 = 0, \quad \forall\, k \geqslant 1,\, N_j = \sum_{k=1}^{j} m_k.$$

We define sequences of sets $(E_k)_{k=1}^{\infty}$, $(G_k)_{k=1}^{\infty}$, by

$$E_k = \Theta_j, \quad G_k = Y_j, \quad \text{if } N_{j-1} < k \leqslant N_j.$$

Then by (3) it comes that

$$\sum_{k=1}^{\infty}|G_k| = \sum_{k=1}^{\infty}|Y_j|m_j \geqslant \sum_{j=1}^{\infty} j2^{j-1}|\Theta_j|m_j = \infty.$$

By the lemma of random translation 1.33, there are translations $(\tau_k)_{k\geqslant 1}$ such that almost every $x \in \mathbb{R}^2$ lies in infinitely many of the translated sets $\tau_k G_k$. Let $j \in \mathbb{N}^*$. For $N_{j-1} < k \leqslant N_j$ define

$$f_k : x \in \mathbb{R}^2 \longmapsto \alpha_j 2^j \chi_{\tau_k E_k}(x)$$

and

$$f : x \in \mathbb{R}^2 \longmapsto \sup_{k\in\mathbb{N}^*} f_k(x).$$

Let $Q = [0,1]^2$ be the unit square.

$$\begin{aligned}\int_Q \Phi(f(x))\mathrm{d}x &\leqslant \sum_{k=1}^{\infty}\int_Q \Phi(f_k(x))\mathrm{d}x \\ &= \sum_{j=1}^{\infty}\sum_{k=N_{j-1}+1}^{N_j}\int_Q \Phi\left(\alpha_j 2^j \chi_{\tau_k E_k}(x)\right)\mathrm{d}x \\ &\leqslant \sum_{j=1}^{\infty}\sum_{k=N_{j-1}+1}^{N_j}\int_{\tau_k E_k}\Phi(\alpha_j 2^j)\mathrm{d}x \\ &= \sum_{j=1}^{\infty}\Phi(\alpha_j 2^j)|\Theta_j| m_j.\end{aligned}$$

Then by (4), $f \in \Phi(L)(\mathbb{R}^2)$. It remains to show that $\mathcal{B}$ doesnt't differentiates the integral of f. We know that almost every $x \in \mathbb{R}^2$ lies in infinitely many of the sets $\tau_k G_x$. The basis $\mathcal{B}$ is invariant by translations, so $\tau_k E_k$ and $\tau_k G_k$ satisfies the last point of lemma 2.20. Hence, for $x \in \tau_k G_k$ with $N_{j-1} < k \leqslant N_j$, there exists a rectangle $R \in \widetilde{\mathcal{B}}^*(x)$ such that

$$\frac{1}{|R|}\int_R f(y)\mathrm{d}y \geqslant \frac{1}{|R|}\int_R f_k(x) = \frac{\alpha_j 2^j |R \cap \tau_k E_k|}{|R|} \geqslant \alpha_j.$$

Since $\alpha_j \longrightarrow \infty$ as $j \to \infty$ and (so k as well), we conclude that $\overline{D}_{\widetilde{\mathcal{B}}^*}(f,x) = \infty$. It's clear that

$$\overline{D}_{\mathcal{B}}(f,x) \geqslant \frac{1}{4}\overline{D}_{\widetilde{\mathcal{B}}^*}(f,x),$$

therefore $\mathcal{B}$ doesn't differentiates $o\left(L\log^+ L(\mathbb{R}^2)\right)$.

Let's move on the second part of the theorem. First of all, let's recall the following fact. Given $\mathcal{A}$, $\mathcal{B}$ two bases, we say that $\mathcal{A}$ is regular with respect to $\mathcal{B}$ if

$$\forall S \in \mathscr{F}, \exists B \in \mathcal{B}: \quad S \subset B, |B| \lesssim |S|.$$

If $\mathcal{A}$ is regular with respect to $\mathcal{B}$ and $\mathcal{B}$ differentiates $L^1(\mathbb{R}^n)$, then so is $\mathcal{A}$. Indeed, for $A \in \mathcal{A}$, if B is the set given by the definition of regularity, there exists a constant c such that

$$\frac{1}{|A|}\int_A |f| \leqslant \frac{c}{|B|}\int_B |f|$$

and

$$M_{\mathcal{A}}f \leqslant cM_{\mathcal{B}}f$$

for all $f \in L^1(\mathbb{R}^n)$. By the Sawyer-Stein principle, we obtain for all $\lambda > 0$

$$|\{x \in \mathbb{R}^n : M_{\mathcal{A}}f(x) > \lambda\}| \leqslant \left|\left\{x \in \mathbb{R}^n : M_{\mathcal{B}}f(x) > \frac{\lambda}{c}\right\}\right| \lesssim \frac{\|f\|_1}{\lambda},$$

and so $\mathcal{A}$ differentiates $L^1(\mathbb{R}^n)$. We get back to the proof of the theorem. We assume that $\mathcal{B}$ fails property (S). Then there exists $\varepsilon_0 > 0$ and $k \in \mathbb{N}$ such that

$$\begin{aligned}\forall (R_1,\dots,R_k) \subset \mathcal{B}^* : \quad &\exists i \neq j : R_i \sim R_j \\ &\exists l \in [\![1,k]\!] : \mathrm{diam}(R_l) \geqslant \varepsilon_0.\end{aligned}$$

We note

$$N := \max\{l \in \mathbb{N}^* : \exists (R_1,\dots,R_l) \subset \mathcal{B}^*, \forall i,j \in [\![1,l]\!], i \neq j \Rightarrow R_i \not\sim R_j\}.$$

If $(R_1, \dots, R_l)$ is a l-tuple with $l \geqslant k$, then there exists $i \neq j$ such that $R_i \sim R_j$. So N is well defined and $N \leqslant k-1$. N is then the size of the biggest finite family of rectangles composed only of pairwise incomparable rectangles. We can have $N = 1$, it means that all the rectangles in $\mathcal{B}$ are comparable. We fix $R_1, \dots, R_N \in \mathcal{B}^*$, N rectangles such that $R_i \not\sim R_j$ whenever $i \neq j$. If R is another rectangle in $\mathcal{B}^*$, then by definition of N there exists an index $i \in [\![1, N]\!]$ such that $R \sim R_i$. Note that the index i isn't necessarily unique because of the lack of transitivity for $\sim$. Then we can write

$$\begin{aligned}\mathcal{B}^* &= \bigcup_{i=1}^{N} \{R \in \mathcal{B}^* : R \sim R_i\} \\ &= \bigcup_{i=1}^{N} \{R \in \mathcal{B}^* : R \sim R_i, \operatorname{diam}(R) \leqslant \varepsilon_0\} \cup \{R \in \mathcal{B}^* : \operatorname{diam}(R) > \varepsilon_0\} \\ &=: \mathcal{B}_1^* \cup E.\end{aligned}$$

The collection E isn't a differentiation basis, so it won't influence the differentiation properties of $\mathcal{B}^*$. However, $\mathcal{B}_1^*$ is a finite collection of bases generetad by monotonic families of rectangles, so by proposition 1.28 (we recall that $\mathcal{B}$ is invariant by translations), $\mathcal{B}_1^*$, and therefore $\mathcal{B}^*$, differentiates $L^1(\mathbb{R}^n)$. Since $\mathcal{B}$ is regular with respect to $\mathcal{B}^*$, by the above argument $\mathcal{B}$ differentiates $L^1(\mathbb{R}^n)$, and the theorem is proven. □

In higher dimensions, the situation is much more complicated. Indeed, the covering behavior of rectangles in $\mathbb{R}^n$ is more complex. In his article, Stokolos nevertheless states that the theorem 2.17 holds for the basis $\mathcal{B}_{n-1} \subset \mathbb{R}^n$. By considering projections on hyperplanes of dimension 2, we come back to the case of the plane.

2.4. Zygmund's conjecture

We conclude this section by presenting another problem concerning rectangles in $\mathbb{R}^n$ whose sides are parallel to the coordinate axis, motivated by theorem 2.10. The idea is to reduce the degree of freedom of the rectangles considered in a different way. Let $1 \leqslant k \leqslant n$. Given n functions ϕ_i

$$\left| \begin{array}{rccc} \phi_i : & \mathbb{R}_+^k & \longrightarrow & \mathbb{R}_+ \\ & (t_1, \dots, t_k) & \longmapsto & \phi_i(t_1, \dots, t_k), \end{array} \right.$$

which are non-decreasing in each variable, we consider the basis $\mathcal{B}_\phi$ in $\mathbb{R}^n$ composed of axis parallel rectangles in $\mathbb{R}^n$ whose side lengths are of the form

$$\phi_1(t_1, \dots t_k) \times \dots \times \phi_n(t_1, \dots, t_k).$$

Of course, if $\phi_i(t_1, \dots, t_k) = t_j$ for some $1 \leqslant j \leqslant k$ and every $i \in \{1, \dots, n\}$, then by Zygmund's theorem (theorem 2.10), the basis $\mathcal{B}_{\phi_1, \dots, \phi_k}$ differentiates $L(\log^+ L)^{k-1}(\mathbb{R}^n)$. From there, Zygmund proposed the following conjecture.

Conjecture 2.21 (Zygmund)**.** The basis $\mathcal{B}_{\phi_1, \dots, \phi_k}$ differentiates $L(\log^+ L)^{k-1}(\mathbb{R}^n)$.

For $k = n$, the basis $\mathcal{B}_{\phi_1, \dots, \phi_k}$ is in fact a subbasis of $\mathcal{I}$, so by theorem 2.3 Zygmund's conjecture is true for $k = n$. For $k = 1$, the conjecture is also true, the basis $\mathcal{B}_{\phi_1}$ differentiates $L^1(\mathbb{R}^n)$. The first non-trivial case is due to Córdoba. In 1978, using geometric argument, Córdoba prove that the Zygmund's conjecture is true for $n = 3$, see [4].

Theorem 2.22 (Córdoba)**.** Let $\phi : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ be function non-decreasing in each variable. Let $\mathcal{B}_\phi$ be the basis composed by rectangles with side parallel to the coordinates axes whose side lengths are of the form

$$s \times t \times \phi(s,t)$$

where $s, t > 0$. Then basis $\mathcal{B}_\Phi$ differentiates $L\log^+ L(\mathbb{R}^3)$.

Example 2.23. The basis composed by rectangles with side parallel to the coordinates axes whose side lengths are of the form

$$s \times t \times \sqrt{st},$$

where $s, t > 0$, differentiates $L\log^+ L(\mathbb{R}^3)$.

However, in general case, the conjecture 2.21 is false. Counter examples are exposed by Soria in [21], or more recently by Rey in [19]. For a basis $\mathcal{B}$ included in $\mathcal{I}_2$, Stokolos's theorem 2.17 asserts that either $\mathcal{B}$ differentiates $L^1(\mathbb{R}^2)$, or $\mathcal{B}$ differentiates $L\log^+ L(\mathbb{R}^2)$. With this result in mind, in 2005 Stokolos proposed another version of the conjecture 2.21, see [26]. To this day, we don't know whether this conjecture holds or not.

Conjecture 2.24 (Zygmund's conjecture, by Stokolos)**.** Let $\mathcal{B} \subset \mathcal{I}$ be a basis invariant by translations. Then, there exists an integer $k \in \{1, \dots, n\}$ such that $\mathcal{B}$ differentiates precisely $L(\log^+ L)^{k-1}(\mathbb{R}^n)$.

3. Sloping rectangles

In this section, instead of considering rectangles with sides parrallel to the axes, we will allow some slopes. We'll consider bases containing rectangles whose sides are not necessarily parallel to the coordinate axes. Of course we stay in the plane, $n = 2$. For a rectangle $R \in \mathcal{R}$, we denote by $\omega_R \in [0, \pi)$ the angle formed by the longest side of rectangle R with the x-axis, see figure 8. If R is a square, there's some ambiguity, but it's not a big deal. Given a set of slopes (also called a set of directions) $\Omega \subset \mathbb{R}$, we define the basis $\mathcal{B}_\Omega \subset \mathcal{R}$ by

$$\mathcal{B}_\Omega := \{R \in \mathcal{R} : \tan(\omega_R) \in \Omega\}.$$

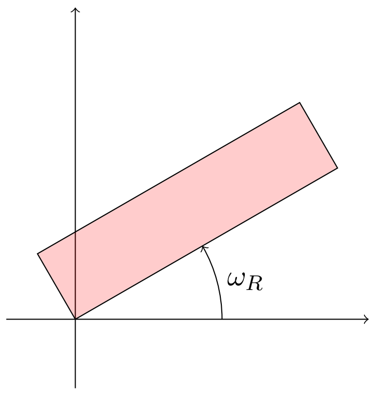

Figure 8: A rectangle in $\mathcal{B}_\Omega$.

Then $\mathcal{B}_\Omega$ is a differentiation basis invariant by homothecies. As usual, we study the maximal operator associated to the basis $\mathcal{B}_\Omega$. Such an operator is called a **directional maximal operator** and is denoted $M_\Omega := M_{\mathcal{B}_\Omega}$. The purpose is to study the behavior of M_Ω according to Ω.

3.1. Kakeya blow

In the following section, we present a geometric concept very usefull in harmonic analysis. For example, this concept was used by Fefferman to disprove the ball multiplier conjecture, see [12]. The main idea is to construct some rectangles such that the size of the union of the dilated rectangles is much bigger than the size of the union of the original rectangles. Of course, we cannot make this construction with any family of rectangles. If we can do it, we say that we have the possibility to make a **Kakeya blow**. Let us formally write this.

Definition 3.1 (Kakeya blow)**.** Let $\mathcal{B} \subset \mathcal{R}$ be a familly of rectangles. We say that we can make a **Kakeya blow** with $\mathcal{B}$ if for any $A > 1$, there exists a finite family of rectangles

$$\{R_i : i \in I_A\} \subset \mathcal{B}$$

such that we have

$$\left| \bigcup_{i \in I_A} 4R_i \right| \geqslant A \left| \bigcup_{i \in I_A} R_i \right|,$$

where $4R_i$ stands for the 4-fold dilation of R_i by its center.

Remark 3.2. The constant A must be large.

Let $\mathcal{B} \subset \mathcal{R}$ be a basis invariant by homothecies, and assume that we can make a Kakeya blow with $\mathcal{B}$. Using the notations from the definition 3.1. We define

$$E_A := \bigcup_{i \in I_A} R_i.$$

Then, for each $x \in 4R_i$, $i \in I_A$, we have

$$M_{\mathcal{B}}\chi_{E_A}(x) \geqslant \frac{1}{|4R_i|}\int_{4R_i} \chi_{E_A}(y)\mathrm{d}y = \frac{|E_A \cap 4R_i|}{|4R_i|} \geqslant \frac{|R_i \cap 4R_i|}{|R_i|} = \frac{1}{16},$$

hence

$$\bigcup_{i \in I_A} 4R_i \subset \left\{ M_{\mathcal{B}}\chi_{E_A} \geqslant \frac{1}{16} \right\}.$$

By the definition of a Kakeya blow, this leads to

$$\left| \left\{ M_{\mathcal{B}}\chi_{E_A} \geqslant \frac{1}{16} \right\} \right| \geqslant A|E_A|$$

for any $A > 1$. By theorem 1.38, this implies that $\mathcal{B}$ is not even a density basis. In summary, the differentiation properties will be bad if we can make a Kakeya blow with it. Now, the question is how do we prove that we can make a Kakeya blow with a given basis $\mathcal{B}$? A first tool is **Perron trees**. The one presented here allows us to deal with all the rectangles of the plane. We shall see later that we can adapt the following construction for other directional bases $\mathcal{B}_\Omega$.

Theorem 3.3 (Perron tree)**.** Let $(T_k)_{1 \leqslant k \leqslant 2^n}$ be the 2^n triangles obtained by joining the points $(0, 1)$ with two successive points in the list $(0, 0), (1, 0), \ldots, (2^n, 0)$. Let $\frac{1}{2} < \alpha < 1$. Then, we can translate horizontally every triangle T_k into a new triangle $\overline{T}_k$ such that

$$\left| \bigcup_{k=1}^{2^n} \overline{T}_k \right| \leqslant \left(\alpha^{2n} + 2(1-\alpha)\right) \left| \bigcup_{k=1}^{2^n} T_k \right|.$$

Proof. The first step is to begin with one triangle T. We cut T in two parts by placing a point on the in the middle of its basis. We obtain two smaller tringles, T_1 and T_2.

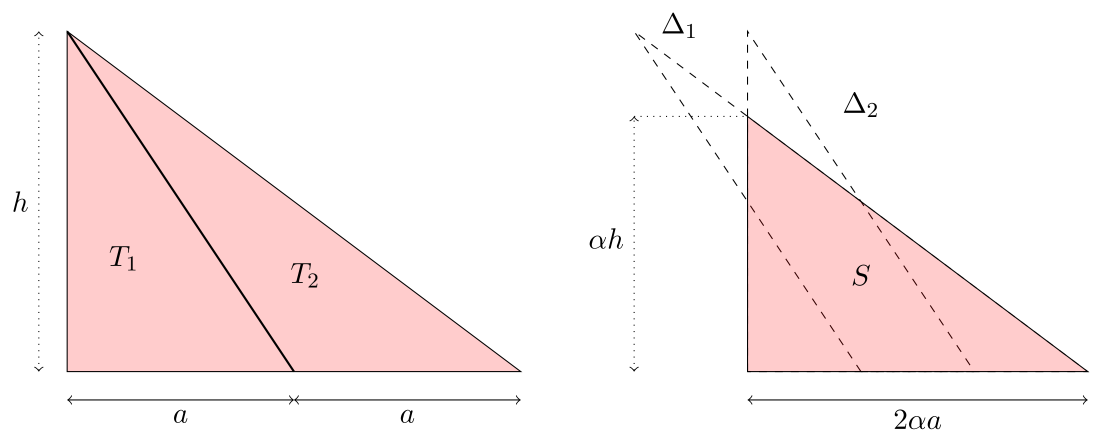

Figure 9: The triangles T_1 and T_2.

We shift T_2 to left such that the non-parallel sides of T_1 and T_2 meet at a point of height αh. We note T_2^* the shifted triangle. The union $T_1 \cup T_2^*$ is then composed by a triangle S and two other smaller triangles Δ_1 and Δ_2, as shown on figure 9. The triangle S is homothetic to $T = T_1 \cup T_2$. Then, it's easy to see that

$$|S| = \alpha^2 |T|$$

and

$$|\Delta_1| + |\Delta_2| = 2(1-\alpha)^2 |T|.$$

Hence,

$$|T_1 \cup T_2^*| = |S| + |\Delta_1| + |\Delta_2| = \left(\alpha^2 + 2(1-\alpha)^2\right)|T|.$$

Now, consider 2^n triangles $(T_i)_{1 \leqslant i \leqslant 2^n}$. There are 2^{n-1} pairs of adjacent triangles (T_1, T_2), ... , (T_{2^n-1}, T_{2^n}).

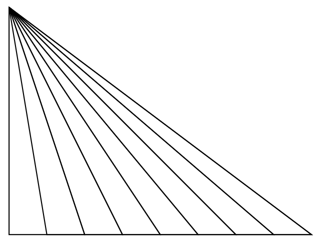

Figure 10: Initialisation of the Perron tree for $n = 3$.

To each pair we apply the preceding construction. We obtain the triangles $S_1, \ldots, S_{2^{n-1}}$ and some excess triangles $(\Delta_k^i)_{i=1,2,\, k=1,\ldots,2^{n-1}}$. We shift S_2 to the left so that it becomes $\widetilde{S}_2$ adjacent to S_1. Similarly, we translate S_3 to the left to a triangle $\widetilde{S}_3$ adjacent to $\widetilde{S}_2$, etc ... The union $S_1 \cup \widetilde{S}_2 \cup \cdots \cup \widetilde{S}_{2^{n-1}}$ is then homothetic the union of our initials triangles $T_1 \cup T_2 \cup \ldots \cup T_{2^n}$, so

$$\left|S_1 \cup \widetilde{S}_2 \cup \cdots \cup \widetilde{S}_{2^{n-1}}\right| = \alpha^2 \left|T_1 \cup T_2 \cup \ldots \cup T_{2^n}\right|.$$

Furthermore, we also have

$$\left|\bigcup \{\text{ shifted excess triangles }\}\right| \leqslant 2(1-\alpha)^2 \left|T_1 \cup T_2 \cup \ldots \cup T_{2^n}\right|.$$

We repeat $n-1$ times this operation with the triangle $S_1 \cup \widetilde{S}_2 \cup \cdots \cup \widetilde{S}_{2^{n-1}}$. We obtain a figure $T_1 \cup \overline{T}_2 \cup \cdots \cup \overline{T}_{2^n}$ composed by a triangle H homothetic to $T_1 \cup T_2 \cup \ldots \cup T_{2^n}$ and some other smaller triangles. We have

$$|H| \leqslant \alpha^{2n} \left|T_1 \cup T_2 \cup \ldots \cup T_{2^n}\right|$$

and the area of the union of the other triangles is less than

$$\begin{aligned}\left(\sum_{k=0}^{n-1} 2\alpha^{2k}(1-\alpha^2)\right) \left|T_1 \cup T_2 \cup \ldots \cup T_{2^n}\right| &\leqslant \frac{2(1-\alpha)^2}{1-\alpha^2} \left|T_1 \cup T_2 \cup \ldots \cup T_{2^n}\right| \\ &\leqslant 2(1-\alpha) \left|T_1 \cup T_2 \cup \ldots \cup T_{2^n}\right|.\end{aligned}$$

Finally, if we let $T_1 = \overline{T}_1$, we have

$$\left|\bigcup_{k=1}^{2^n} \overline{T}_k\right| \leqslant \left(\alpha^{2n} + 2(1-\alpha)\right) \left|\bigcup_{k=1}^{2^n} T_k\right|$$

and the theorem is proven. □

This construction allows us to make a Kakeya blow for $\mathcal{R}$. For $1 \leqslant k \leqslant 2^n$, we construct a rectangle R_k associated to $\overline{T}_k$ as follows. We denote by A_k, P_k, P_{k-1} the vertices of $\overline{T}_k$. We define R_k as the only rectangle whose lower left corner coincides withP_{k-1}, and whose upper right corner M_k is such that

$$\overrightarrow{A_k M_k} = \frac{1}{3}\overrightarrow{A_k P_k}$$

(see figure 11). Actually, the placement of the point M_k isn't very important, we just want $R_k \subset \overline{T}_k$, not too tiny and "aligned with" $\overline{T}_k$. Then, we translate the rectangle R_k along the line $(A_k P_{k-1})$ such the translated rectangle $\widetilde{R_k}$ is such that $\widetilde{R_k} \cap \overline{T}_k$ is a singleton.

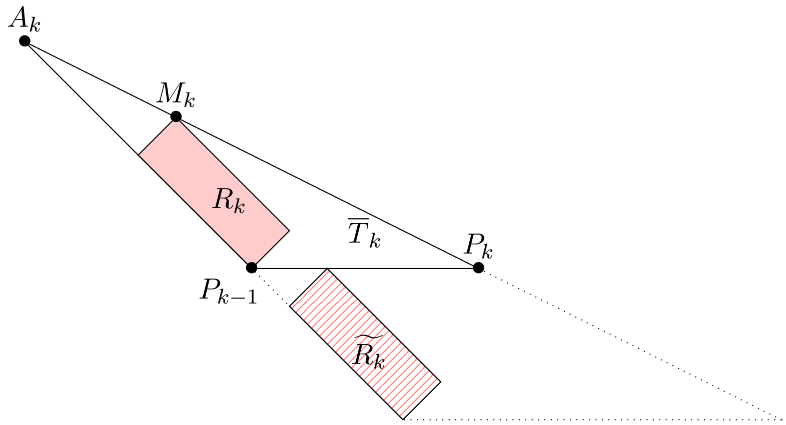

Figure 11: The rectangles R_k and $\widetilde{R_k}$.

The important fact is that the rectangles $\widetilde{R_k}$ are pairwise disjoint, see [12, Lemma 3]. We define $\varepsilon(n,\alpha) := \alpha^{2n} + 2(1-\alpha)$. Since $R_k \subset \overline{T}_k$, we obviously have

$$\bigcup_{k=1}^{2^n} R_k \subset \bigcup_{k=1}^{2^n} \overline{T}_k$$

and

$$\left|\bigcup_{k=1}^{2^n} R_k\right| \leqslant \left|\bigcup_{k=1}^{2^n} \overline{T}_k\right| \leqslant \varepsilon(n,\alpha)\left|\bigcup_{k=1}^{2^n} T_k\right| = \varepsilon(n,\alpha)\sum_{k=1}^{2^n}|T_k|,$$

because the triangles T_k are disjoint. Furthermore, there exists a constant $C > 1$ such that for all $k \in \{1,\dots,2^n\}$, $|T_k| = C|R_k| = C|\widetilde{R_k}|$, so

$$\left|\bigcup_{k=1}^{2^n} R_k\right| \leqslant \varepsilon(n,\alpha)C\sum_{k=1}^{2^n}|\widetilde{R_k}| = \varepsilon(n,\alpha)C\left|\bigcup_{k=1}^{2^n} \widetilde{R_k}\right|$$

because the $(\widetilde{R_k})_{k\in\mathbb{N}^*}$ are pairwise disjoint. In addition, it's clear that for each k we have $\widetilde{R_k} \subset 4R_k$. Let $A \gg 1$ be given. Then, by choosing α close enough to 1 and n very large, is it possible to have

$$\varepsilon(n,\alpha) \leqslant \frac{1}{CA}$$

and

$$\left|\bigcup_{k=1}^{2^n} 4R_k\right| \geqslant A\left|\bigcup_{k=1}^{2^n} R_k\right|.$$

It's then possible to make a Kakeya blow with the basis $\mathcal{R}$, by the above the following corollary is immediate.

Corollary 3.4. $\mathcal{R}$ is not a density basis.

In fact, by reading carefuly the construction of a perron tree, we realize that we haven't used all rectangles in $\mathcal{R}$, but just some of them with particular slopes. For example, the rectangles R_k we just constructed have slopes $\frac{-1}{k}$. Symmetrizing the figure, we have the following.

Theorem 3.5. If $\Omega = \left\{\frac{1}{k} : k \in \mathbb{N}^*\right\}$, then it is possible to make a Kakeya blow with $\mathcal{B}_\Omega$.

Our next purpose is to adapt this construction to deal with some other set of slopes Ω. Before going further with Perron trees, we are going to present a remarkable theorem proven by Bateman in 2009, in [2].

3.2. Bateman's theorem

We start this section by a disclaimer about Bateman's theorem. Recently, in [15], Hagelstein, Radillo-Murguia and Stokolos may have found an error in Bateman's article ([2]). To this day, we don't know whether Bateman's theorem holds, and we don't have a correction for the proof yet. In the following, we consider that Bateman's theorem is true.

Let Ω be a set of slopes. Roughly, Bateman's theorem asserts that either the maximal directional operator M_Ω is always bounded, or it never is. Bateman even gives us a complete characterization of both cases. Before stating the theorem precisely, let's introduce a few definitions.

Definition 3.6. A sequence of real numbers $\{u_k : k \in \mathbb{N}\}$ is said to be **lacunary** converging to $l \in \mathbb{R}$ if there exists a real $\lambda \in (0,1)$ such that

$$|u_{k+1} - l| \leqslant \lambda |u_k - l|$$

for eack $k \in \mathbb{N}$.

Examples 3.7. The sequences $\{2^{-k} : k \in \mathbb{N}\}$, $\{\frac{1}{k!} : k \in \mathbb{N}\}$ are lacunary (and converging to 0).

Theorem 3.8. Let Ω be a lacunary sequence converging to 0. Then M_Ω is bounded on $L^p(\mathbb{R}^2)$ for any $p > 1$.

In 1977, in [6], Córdoba and Fefferman proved that the basis $\mathcal{B}_\Omega$ has the covering property (V_2) (see [5]), and so that M_Ω is bounded on $L^2(\mathbb{R}^2)$. One year later, in 1978, Nagel, Stein and Wainger used Fourier analysis tecnhiques to show that is in fact M_Ω is bounded on L^p for any $p > 1$ (see [17] for details). This result is the starting point for the result of Bateman. We now introduce the notion of lacunary set (of finite order).

Definition 3.9 (Lacunary set of finite order)**.** The lacunary sets are defined recursively.

- A lacunary set of order 0 in $\mathbb{R}$ is a set which is either empty or a singleton.
- Let $N \in \mathbb{N}^*$. We say that a set $\Omega \subset \mathbb{R}$ is lacunary of order at most $N+1$ if there exists a lacunary sequence L such that : for any $a, b \in L$ with $a < b$ and $(a,b) \cap L = \emptyset$, the set $\Omega \cap (a,b)$ is a lacunary set of order at most N.

Example 3.10. The set

$$\Omega = \left\{ \frac{1}{4^l} + \frac{1}{2^k} : k, l \in \mathbb{N}, k \geqslant l \right\}$$

is a lacunary set of order 2.

Definition 3.11 (Finitely lacunary set)**.** A set $\Omega \subset \mathbb{R}$ is said to be **finitely lacunary** if we can recover Ω by a finite number of lacunary set of finite order, *i.e.* if there exists $\Omega_1, \dots, \Omega_m$ which are lacunary of finite orders $N_1, \dots, N_m$ such that

$$\Omega \subset \bigcup_{k=1}^{m} \Omega_k.$$

We can now enunciate the theorem of Bateman. We refer to [2] for the proof.

Theorem 3.12 (Bateman, 2009)**.** Let Ω be a set of directions, and $1 < p < \infty$. The following are equivalent.

(i) The operator M_Ω is bounded on L^p.

(ii) The set Ω is finitely lacunary.

(iii) It's not possible to make a Kakeya blow with $\mathcal{B}_\Omega$.

Remark 3.13. In his article, Bateman wrote his theorem for $1 < p < \infty$, but in fact the result holds for $p = \infty$. We can read [8, p5] for some explanations.

Hence, given a set of directions Ω, we have the following alternative :

- The operator M_Ω is bounded on L^p for all $1 < p < \infty$. In this case, we will say that Ω is a **good set** of directions.
- The operator M_Ω is unbounded on L^p for all $1 < p < \infty$, and we will say that Ω is a **bad set** of directions.

Our goal is now to know whenever a given set Ω is a bad set or a good set. From the above, we have our first examples of bad and good sets of directions.

Examples 3.14.

- The set $\Omega = \left\{\frac{1}{k} : k \in \mathbb{N}^*\right\}$ is a bad set of directions.
- The set $\Omega = \left\{2^{-k} : k \in \mathbb{N}\right\}$ is a good set of directions.
- The set $\Omega = \left\{\frac{1}{k!} : k \in \mathbb{N}\right\}$ is a good set of directions.

Unfortunatelly, the characterization given by Bateman, although comprehensive, is sometimes very complicated to apply in practice. For example, it seems diffucult to show that $\left\{\frac{1}{k} : k \in \mathbb{N}^*\right\}$ is not finitely lacunary.

3.3. Generalized Perron trees

We are going to generalize the Perron tree construction we saw earlier. The above construction deals only with rectangles with slopes in $\{1/k : k \in \mathbb{N}^*\}$. The purpose here is to deal with set of slopes of the form

$$\Omega = \left\{\frac{-1}{u_k} : k \in \mathbb{N}^*\right\}$$

where $\{u_k\}_{k \in \mathbb{N}^*}$ is a positive stricly increasing sequence (the minus is just here to be coherent with the figures, but it's useless and will be dropped). The idea is to adapt the previous construction to the triangles $T_k = AU_kU_{k-1}$, where $A := (0, 1)$ and $U_k := (u_k, 0)$. However, there's a price to pay, there's no guarantee that the construction can be carried out for any sequence $(u_k)_{k \in \mathbb{N}^*}$.

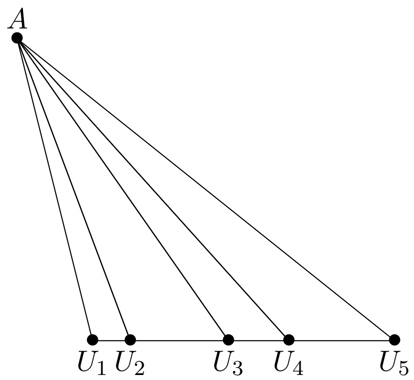

Figure 12: The triangles T_k, $k \in \{2,3,4,5\}$.

Compared to the original Perron tree, the main difference is that the base lengths of the triangles vary. Let us explain what are the consequences for two triangles. Let T_a, T_b be two adjacent triangles whose base sizes are respectively a and b, see figure 13.

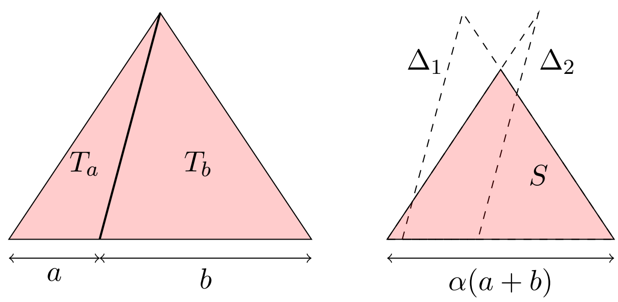

Figure 13: The triangles T_a and T_b.

Let S be the little triangle homothetic to $T_a \cup T_b$ (the red one on the right), and Δ_1, Δ_2 be the excess triangles. We have

$$|S| = \alpha^2 |T_a \cup T_b|,$$

$$|\Delta_1 \cup \Delta_2| = (1-\alpha)^2 \left(\frac{a}{b} + \frac{b}{a}\right) |T_a \cup T_b|.$$

Of course, if $a = b$, we come back to the original construction. For each step of the construction, we have to ensure that the quanity $\left(\frac{a}{b} + \frac{b}{a}\right)$ remains bounded. As in the original Perron tree, the idea is now to execute this operation with many pairs of triangles. At the end of the day, using the previous notations, we obtain

$$\left|\bigcup_{k=1}^{2^n} \overline{T}_k\right| \leqslant \left(\alpha^{2n} + (1-\alpha)H_n\right) \left|\bigcup_{k=1}^{2^n} T_k\right|$$

where H_n is the maximum of the quantities $\left(\frac{a}{b} + \frac{b}{a}\right)$ after n steps in the construction. Then we have to understand how the quantity H_n behaves, for example under what conditions it remains bounded. Let's look at what we obtain after two steps. If we consider four triangles with length basis a, b, c and d, see figure 14, then the new error term is

$$\frac{a+b}{c+d} + \frac{c+d}{a+b}.$$

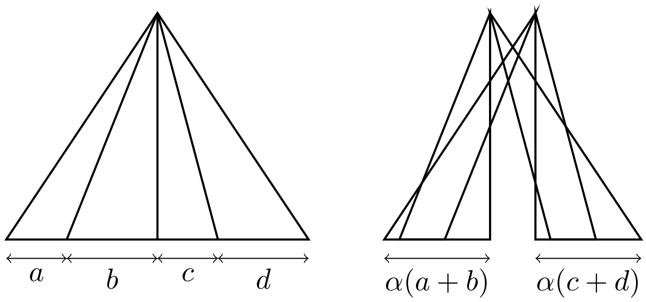

Figure 14: Generalized Perron tree, $n = 2$.

If we generalize, at step l in the construction, the quantity $\left(\frac{a}{b}+\frac{b}{a}\right)$ is then of the form

$$\frac{\sum_{i=k+1}^{k+2^l}(u_i-u_{i-1})}{\sum_{i=k+2^l+1}^{k+2^{l+1}}(u_i-u_{i-1})}+\frac{\sum_{i=k+2^l+1}^{k+2^{l+1}}(u_i-u_{i-1})}{\sum_{i=k+1}^{k+2^l}(u_i-u_{i-1})}=\frac{u_{k+2^l}-u_k}{u_{k+2^{l+1}}-u_{k+2^l}}+\frac{u_{k+2^{l+1}}-u_{k+2^l}}{u_{k+2^l}-u_k}$$

for some appropriate $k \in \mathbb{N}$ such that $k + 2^l \leqslant 2^n$ and $k \geqslant 2^l$ (roughly, k represents the group of 2^l triangles chosen). This observation motivates the following definition.

Definition 3.15 (Perron factor). Assume that $\Omega = \left(u_k^{-1}\right)_{k\in\mathbb{N}^*}$, where $(u_k)_{k\in\mathbb{N}^*} \subset \left(0, \frac{\pi}{2}\right)$ is an increasing set of slopes. We define the **Perron factor** of Ω as

$$PF(\Omega) := \sup_{\substack{k\geqslant 1\\ j\leqslant k}}\left(\frac{u_{k+2j}-u_{k+j}}{u_{k+j}-u_k}+\frac{u_{k+j}-u_k}{u_{k+2j}-u_{k+j}}\right)\in[2,\infty].$$

Remark 3.16. In the definition, the index j corresponds to the index 2^l in our calculation.

Examples 3.17.

- If $\Omega = \left\{\frac{1}{k} : k \in \mathbb{N}^*\right\}$, then $PF(\Omega) = 2$.
- If $\Omega = \left\{2^{-k} : k \in \mathbb{N}\right\}$, then $PF(\Omega) = \infty$.

In [16], Hare and Rönning proved the following theorem, which is a **quantitative** sufficient condition on Ω to be able to make a Kakeya blow with $\mathcal{B}_\Omega$.

Theorem 3.18. Assume that $\Omega = \left(u_k^{-1}\right)_{k\in\mathbb{N}^*}$, where $(u_k)_{k\in\mathbb{N}^*} \subset \left(0, \frac{\pi}{2}\right)$ is an increasing set of slopes. Assume that

$$PF(\Omega) < \infty.$$

Then it's possible to make a Kakeya blow with $\mathcal{B}_\Omega$.

For example, we can apply the theorem with example 3.17. Because of Bateman's theorem (if this last holds), theorem 3.18 has many formulations. Indeed, we have the following equivalences.

$$\begin{aligned} PF(\Omega) < \infty &\Rightarrow \text{It's possible to make a Kakeya blow with } \mathcal{B}_\Omega,\\ &\Leftrightarrow \|M_\Omega\|_p = \infty \text{ for any } 1 < p \leqslant \infty,\\ &\Leftrightarrow \Omega \text{ is not finitely lacunary},\\ &\Leftrightarrow \ \Omega \text{ is a bad set of directions.}\end{aligned}$$

The main problem with the Perron factor and its application is that we need some regularity on the set of directions, the sequence $(u_k)_{k\in\mathbb{N}^*}$ must be increasing. For example, what happens if we try to apply this result for sets of directions such that

$$\Omega_{\mathrm{sin,lin}} := \left\{ \frac{\sin(k)}{k} : k \in \mathbb{N}^* \right\}, \quad \Omega_{\mathrm{sin,lac}} := \left\{ \frac{\sin(k)}{2^k} : k \in \mathbb{N}^* \right\}.$$

(we are using the notation of Gauvan's PhD thesis, see [13]). Because of the oscillations of the trigonometric functions, it seems complicated to write explicitly these sequences as the form used in the definition 3.15. What we are looking for is a similar result valid for all set of slopes $\Omega \subset \mathbb{R}$. Such a result was proven by D'Aniello, Gauvan and Moonens, see [7] or [13]. They introduce a more precise quantity than the Perron factor, the **Perron capacity**.

Definition 3.19. Let $\Omega \subset \mathbb{R}$ be a set of directions. We define the **Perron capacity** of Ω as

$$PC(\Omega) := \liminf_{N\to\infty} \inf_{\substack{U\subset\Omega \\ \#U=2^N}} PF(\Omega) \in [2, \infty].$$

Then they proved another quantitative and sufficient condition, usable without restriction on the set Ω.

Theorem 3.20 (D'Aniello, Gauvan, Moonens)**.** Let $\Omega \subset \mathbb{R}$ be a set of directions. Assume that we have

$$PC(\Omega) < \infty.$$

Then it's possible to make a Kakeya blow with $\mathcal{B}_\Omega$.

In the same paper [7], the three authors used theorem 3.20 to deal with the case of $\Omega_{\mathrm{sin,lin}}$. They obtained the following, and its immediate corollary.

Theorem 3.21. The Perron capacity of $\Omega_{\mathrm{sin,lin}}$ is finite,

$$PC(\Omega_{\mathrm{sin,lin}}) < \infty.$$

Corollary 3.22. The set $\Omega_{\mathrm{sin,lin}}$ is a bad set of directions.

Remark 3.23. Of course the result holds for $\Omega_{\mathrm{cos,lin}} := \{\cos(k)/k : k \in \mathbb{N}^*\}$.

The case of $\Omega_{\mathrm{sin,lac}}$ is still open. However, if we replace the sin by a sequence $(X_k)_{k\in\mathbb{N}^*}$, where the X_k are random variables uniformly distributed in $(0, 1)$, and independant, then, almost surely, the set

$$\Omega_{\mathrm{rand,lac}} := \left\{ \frac{X_k}{2^k} : k \in \mathbb{N}^* \right\}$$

is a bad set of directions. Indeed, in his PhD thesis [13], Gauvan proved the following.

Theorem 3.24. The Perron capacity of $\Omega_{\mathrm{rand,lac}}$ is finite almost surely, *i.e.* almost surely we have

$$PC(\Omega_{\mathrm{rand,lac}}) < \infty.$$

Remark 3.25. The same result is also true for $\Omega_{\mathrm{rand,lin}}$.

3.4. The almost-orthogonality principle

The above gives us a sufficiant condition to be able to make a Kakeya blow. Here, we focus on the opposite. Is there a condition to ensure that we cannot make a Kakeya blow (and thus assert that a set is a good set) ? One possible answer is the almost-orthogonality principle.

Given a set of slopes Ω, the idea is to break Ω into disjoint blocks and to study our maximal operator on these separated blocks. We start by introducing some notations. For the rest of this section, Ω will be a set of directions included in $[0, \pi/4)$, and let $\Omega_0 \subset \Omega$ be an ordered subset of Ω, *i.e.* $\Omega_0 = \{\theta_k\}_{k\in\mathbb{N}^*}$ with

$$\theta_1 > \theta_2 > \cdots > \theta_n > \cdots$$

The choice of Ω_0 will be very important, it will determine how do we break the original set Ω into smaller blocks. Furthermore, we will see that M_Ω must be bounded on Ω_0, otherwise the theorem would be empty. Then we define for $k \in \mathbb{N}^*$

$$\Omega_k := \{\omega \in \Omega : \theta_k \leqslant \omega < \theta_{k-1}\} = \Omega \cap [\theta_k, \theta_{k-1})$$

with $\theta_0 = \pi/4$. We assume that we choose Ω_0 such that

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k. \tag{5}$$

In [1], Alfonseca, Soria and Vargas proved the following.

Theorem 3.26 (Alfonseca, Soria, Vargas)**.** There exists a constant C such that

$$\|M_\Omega\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leqslant \sup_{k\geqslant 1} \|M_{\Omega_k}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} + C\|M_{\Omega_0}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}.$$

Remark 3.27. Because of Bateman's theorem, to study the differentiation properties of $\mathcal{B}_\Omega$ it's enough to look at the $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ norm of M_Ω : if M_Ω is bounded on $L^2(\mathbb{R}^2)$, then M_Ω. is bounded on $L^p(\mathbb{R}^2)$ for all $p > 1$.

Let's explain a way to use this theorem. Imagine that Ω_0 is a lacunary sequence, for example $\Omega_0 = \{2^{-k}\}_{k\in\mathbb{N}^*}$. By adding this sequence to our set Ω, we can assume that (5) holds. Then by the theorem 3.8 of Córdoba and Fefferman, we know that

$$\|M_{\Omega_0}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} < \infty.$$

Then it comes that

$$\|M_\Omega\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \lesssim \sup_{k\geqslant 1} \|M_{\Omega_k}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}.$$

We can go further. A way to ensure that the quantity $\sup_{k\geqslant 1} \|M_{\Omega_k}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}$ is finite is to have an uniform bound for the sequence $(\#\Omega_k)_{k\in\mathbb{N}^*}$. The following proposition sums it up.

Proposition 3.28. Let $\Omega \subset [0, \pi/4)$. Assume that there exists a constant C such that for each $k \in \mathbb{N}^*$

$$\#\left\{\omega \in \Omega : \omega \in [2^{-k-1}, 2^{-k}]\right\} \leqslant C.$$

Then Ω is a good set of directions.

We already talked about the case of $\Omega_{\text{sin,lac}}$. Can we use the theorem 3.26 for it ? By the proposition 3.28, if there exists a constant C such that

$$\#\left\{k \in \mathbb{N} : \frac{\sin(k)}{2^k} \in [2^{-n-1}, 2^{-n}]\right\} \leqslant C$$

for each $n \in \mathbb{N}^*$, then $\Omega_{\text{sin,lac}}$ will be a good set. We note for $n \in \mathbb{N}^*$

$$a_n := \#\left\{k \in \mathbb{N} : \frac{\sin(k)}{2^k} \in [2^{-n-1}, 2^{-n}]\right\} \quad \text{and} \quad b_n := \sum_{k=1}^{n} a_k.$$

One has

$$b_n = \#\left\{k \in \mathbb{N} : \frac{\sin(k)}{2^k} \in [2^{-n-1}, 2^{-1}]\right\} = \#\left\{k \in [\![1, n]\!] : \frac{\sin(k)}{2^k} \geqslant 2^{-n-1}\right\}.$$

We conjecture the following result.

Conjecture 3.29. There exists a constant $C > 0$ such that for all $n \in \mathbb{N}^*$

$$\frac{n}{2} - C \leqslant b_n \leqslant \frac{n}{2} + C.$$

By writting $a_n = b_n - b_{n-1}$, it is clear that the conjecture implies the desired result. To this day, we don't know if this conjecture holds or not. Nevertheless, we can prove the following weaker result

Proposition 3.30. One has $b_n \sim \frac{n}{2}$.

Proof. For $n \in \mathbb{N}^*$ and $x \in [0, 1]$, we denote

$$F_n(x) = \{k \in [\![1, n]\!] : \sin(k) \geqslant x\}$$

and $f_n(x) = \text{card} F_n(x)$. By applying the ergodic theorem to the ergodic application

$$T : \begin{array}{ccc} \mathbb{R}/2\pi\mathbb{Z} & \longrightarrow & \mathbb{R}/2\pi\mathbb{Z} \\ x & \longmapsto & x+1 \end{array}$$

it comes that for $x \in [0, 1]$

$$\frac{f_n(x)}{n} \underset{n\to\infty}{\longrightarrow} \frac{\pi - 2\arcsin(x)}{2\pi} = \frac{1}{2} - \frac{\arcsin(x)}{\pi}$$

(recall that T is ergodic because $1 \notin 2\pi\mathbb{Q}$).
Let $B_n := \left\{k \in [\![1, n]\!] : \frac{\sin(k)}{2^k} \geqslant 2^{-n-1}\right\}$. Clearly, we have $B_n \subset F_n(0)$, so $b_n \leqslant f_n(0)$, and

$$\limsup_{n\to\infty} \frac{b_n}{n} \leqslant \frac{1}{2}.$$

Let $N \geqslant 2$. Note that for $n \geqslant N - 2$,

$$F_n(2^{-N}) \backslash \{n, n-1, \ldots, n-N+2\} \subset B_n.$$

Indeed, if $\sin(k) \geqslant 2^{-N}$ and $k \leqslant n - N + 1$ then

$$\frac{\sin(k)}{2^k} \geqslant \frac{1}{2^{N+k}} \geqslant \frac{1}{2^{n+1}},$$

so $k \in B_n$. Thus,

$$f_n(2^{-N}) - (N-1) \leqslant b_n,$$

and again using the ergodic theorem, it follows that

$$\liminf_{n\to\infty} \frac{b_n}{n} \geqslant \liminf_{n\to\infty} \frac{f_n(2^{-N}) - (N-1)}{n} = \frac{1}{2} - \frac{\arcsin(2^{-N})}{\pi}.$$

Thus, for all $N \geqslant 2$, we have

$$\frac{1}{2} - \frac{\arcsin(2^{-N})}{\pi} \leqslant \liminf_{n\to\infty} \frac{b_n}{n} \leqslant \limsup_{n\to\infty} \frac{b_n}{n} \leqslant \frac{1}{2}.$$

Since $\arcsin(2^{-N})$ converges to 0 wh en $N \longrightarrow \infty$, we have

$$\frac{1}{2} \leqslant \liminf_{n\to\infty} \frac{b_n}{n} \leqslant \limsup_{n\to\infty} \frac{b_n}{n} \leqslant \frac{1}{2}$$

and

$$b_n \sim \frac{n}{2}.$$

□

Another application of the almost-orthogonality principle is a simple proof of the Katz conjecture.

Corollary 3.31. Let $\Omega \subset [0, \pi/4)$ with $\#\Omega = N > 1$. Then M_Ω is of strong type $(2,2)$ and

$$\|M_\Omega\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \lesssim \log N, \tag{6}$$

where the implicit constant is independant of Ω.

Proof. The fact that M_Ω is of strong type (2,2) if Ω is finite is already known. Let Ω be a set of directions of cardinality $N > 1$. We first assume that $N = 2^M$, $M \geqslant 1$, and we will prove the theorem by induction on M. For $M = 1$, the result is already known. Assume that $M \geqslant 2$. We suppose that we have 6 for all sets of cardinal 2^k, where $1 \leqslant k < M$, and we note K the constant. We order the elements of Ω, and note

$$\Omega = \{\omega_1 > \omega_2 > \cdots > \omega_N\}.$$

We define the set Ω_0 as $\Omega_0 := \{\omega_{N/2}, \omega_N\}$. Then we have $\Omega_1 = \{\omega_1, \ldots, \omega_{N/2}\}$ and $\Omega_2 = \{\omega_{N/2+1}, \ldots, \omega_N\}$, and $\#\Omega_1 = \#\Omega_2 = \frac{N}{2}$ Then, by theorem 3.26, we have

$$\|M_\Omega\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leqslant K \log \frac{N}{2} + 2C = K \log N - K \log 2 + 2C.$$

So, by choosing K such that $K \geqslant \frac{2C}{\log 2}$, we have the result for $N = 2^M$.
Now, if N is not a power of 2. There exists an integer $M \geqslant 2$ such that $2^{M-1} < N < 2^M$. Let Ω' be set a of cardinal 2^M such that $\Omega \subset \Omega'$. Since $\|M_\Omega\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leqslant \|M_{\Omega'}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}$ and $\#\Omega'$ is a power of 2, we have

$$\|M_\Omega\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leqslant K \log 2^M = \frac{M}{M-1} K \log 2^{M-1} \leqslant 2K \log N.$$

The result is then true for the constant $K' = 2K$. □

A. **Orlicz spaces**

Here we present the basic knowledge we need about Orlicz space. We need these spaces to generalize Lebesgue spaces $L^p(\mathbb{R}^n)$ and to find intermediate spaces between them. Proofs, details, and further results about this topic can be found in [18].

Definition A.1 (Orlicz function)**.** An **Orlicz function** is a function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ such that :

- Φ is convex, continuous and increasing.
- $\Phi(0) = 0$.
- $\Phi(t) \longrightarrow \infty$ as $t \longrightarrow \infty$.

Examples A.2. The following are Orlicz functions.

- $\Phi : t \longmapsto \frac{t^p}{p}$, for $1 \leqslant p < \infty$.
- $\Phi : t \longmapsto t(1 + \log^+ t)$.
- $\Phi : t \longmapsto e^t - 1$.

Definition A.3. Given Φ be an Orlicz function, we note $L^\Phi(\mathbb{R}^n)$ the **Orlicz space** the set :

$$L^\Phi(\mathbb{R}^n) := \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(|f|) < \infty \right\}.$$

Examples A.4.

- If $\Phi(t) = \frac{t^p}{p}$, $1 \leqslant p < \infty$, then $L^\Phi(\mathbb{R}^n)$ is the Lebesgue space $L^p(\mathbb{R}^n)$.
- If $\Phi(t) = t(1 + \log^+ t)$, the Orlicz space $L^\Phi(\mathbb{R}^n)$ is denoted $L \log^+ L(\mathbb{R}^n)$.

We fix an Orlicz function Φ. For $f \in L^\Phi(\mathbb{R}^n)$, we define the quantity

$$\|f\|_\Phi := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f|}{\lambda}\right) \leqslant 1 \right\}.$$

Then f is in $L^\Phi(\mathbb{R}^n)$ if and only if $\|f\|_\Phi < \infty$. Thanks to this, we have an interesting structure on $L^\Phi(\mathbb{R}^n)$.

Theorem A.5. The space $(L^\Phi(\mathbb{R}^n), \|\cdot\|_\Phi)$ is a Banach space.

We also define the notion of weak-type (Φ,Φ).

Definition A.6. Let T be an operator mapping $L^\Phi(\mathbb{R}^n)$ into $\mathcal{M}(\mathbb{R}^n)$. We say that T is

- of strong-type (Φ,Φ) if there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} \Phi(|Tf|) \leqslant C \int_{\mathbb{R}^n} \Phi(|f|).$$

- of weak-type (Φ,Φ) if there exists a constant $C > 0$ such that

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leqslant C \int_{\mathbb{R}^n} \Phi\left(\frac{|f|}{\lambda}\right).$$

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